Subgraph Deletion of 4-Regular Graphs and Their Genus Ranges

Hayden Hunter

University of South Florida

December 7, 2018

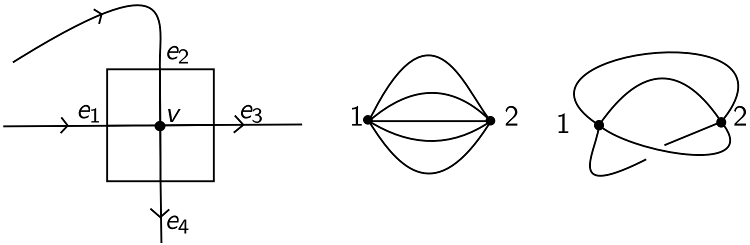
Definition

Let A be an alphabet. A *double occurence word* over A is a word which contains each symbol of A exactly 0 or 2 times. We denote the set of double occurence words as A_{DOW}.

Example: Let $A = \mathbb{N}$. Then $121323 \in A_{DOW}$

Definition: Rigid Vertex

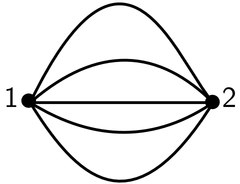
1. Rotation System
2. Sharp Corners are not permitted

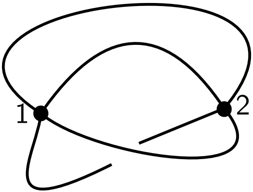


Definition: Assembly Graph

Definition: Assembly Graph

Let Γ be a graph where each vertex is a rigid vertex of degree 4 or 1. Then we call Γ an *assembly graph*.

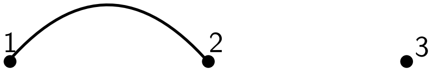




Definition: Rigid Vertex and Assembly Graph

The graph below is the assembly graph of 121323. For $\Gamma = (V(\Gamma), E(\Gamma))$ we have that

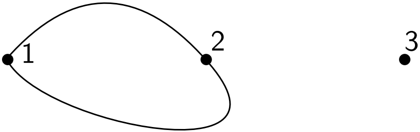
$$1 \to 2$$



Definition: Rigid Vertex and Assembly Graph

The graph below is the assembly graph of 121323. For $\Gamma = (V(\Gamma), E(\Gamma))$ we have that

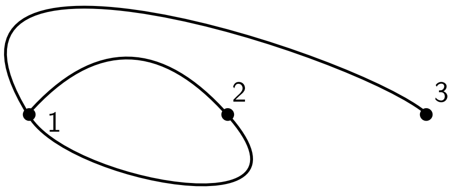
$$1 \to 2 \to 1$$



Definition: Rigid Vertex and Assembly Graph

The graph below is the assembly graph of 121323. For $\Gamma = (V(\Gamma), E(\Gamma))$ we have that

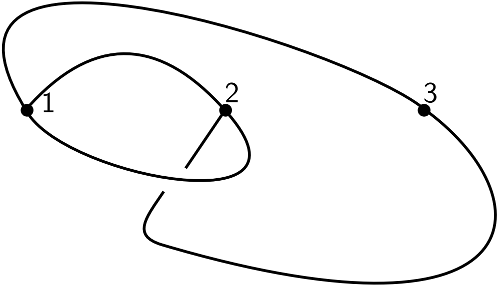
$$1 \to 2 \to 1 \to 3$$



Definition: Rigid Vertex and Assembly Graph

The graph below is the assembly graph of 121323. For $\Gamma = (V(\Gamma), E(\Gamma))$ we have that

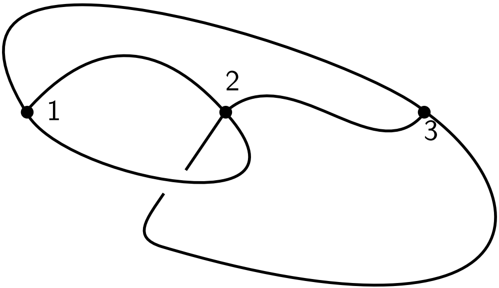
$$1 \to 2 \to 1 \to 3 \to 2$$



Definition: Rigid Vertex and Assembly Graph

The graph below is the assembly graph of 121323. For $\Gamma = (V(\Gamma), E(\Gamma))$ we have that

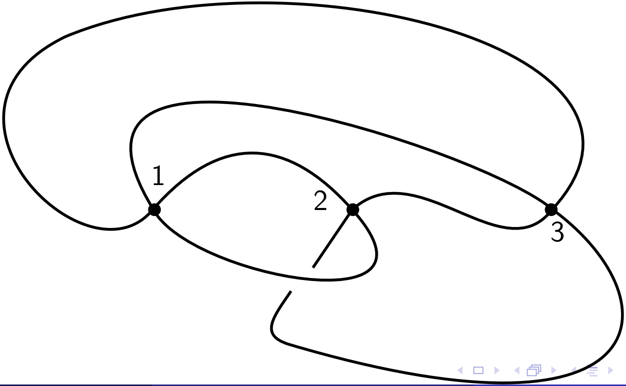
$$1 \to 2 \to 1 \to 3 \to 2 \to 3$$



Definition: Rigid Vertex and Assembly Graph

The graph below is the assembly graph of 121323. For $\Gamma = (V(\Gamma), E(\Gamma))$ we have that

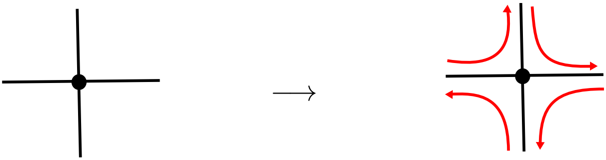
$$1 \to 2 \to 1 \to 3 \to 2 \to 3 \to 1$$

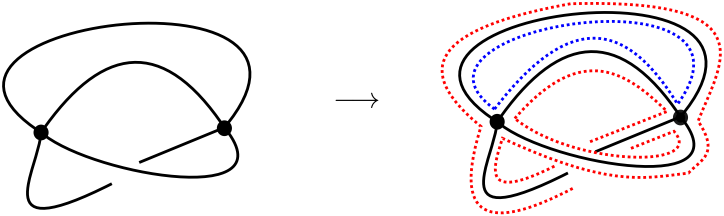


Graphs and Their Genus

- Ribbon graphs and boundary components
- The Euler Characteristic of these ribbon graphs
- The change of a boundary connection at a vertex

Ribbon Graphs





Euler Characteristic of Ribbon Graphs

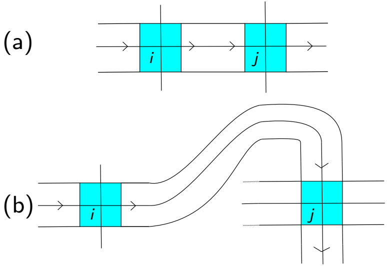
Definition

The integer g that represents the number of handles a topological space has is the *genus* of that topological space

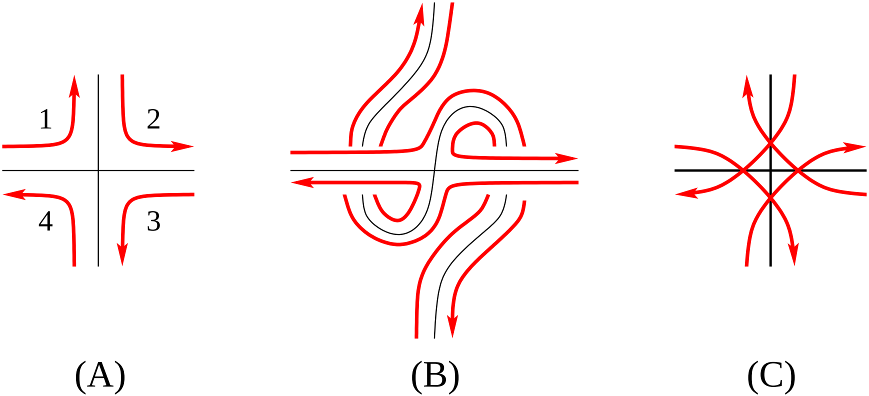
- The genus of an assembly graph Γ is the minimum number of handles of the topological space Γ can be embedded into.
- Letting $b(\Gamma)$ be the number of boundary components, and considering that each assembly graph with n vertices has $2n$ edges,

$$\chi = n - 2n + b(\Gamma) \implies g(\Gamma) = \frac{1}{2}(2 + n - b(\Gamma))$$

For each vertex there are two possible ways to construct our ribbon graph at the second occurrence.



Thus we have that for an assembly graph Γ with n vertices, that there exists 2^n boundary connections and thus 2^n different graphical embeddings.


1
2
3
4
(A)
(B)
(C)

Definition: Genus Range

Letting a be the minimum genus of Γ and b being the maximum genus of Γ, we say that the *genus range* of Γ is $[a, b]$.

- The minimum genus will be the minimum number of handles of the topological space that Γ is being embedded onto.
- The maximum genus will be the maximum number of handles of the topological space that Γ is being embedded onto.

Let $u = u_1 u_2 \dots u_n$. Then we say that the reverse of u denoted as $u^R = u_n u_{n-1} \dots u_2 u_1$.

Definition: Repeat and Return Insertions

Let $w = xyz \in A$. Let u be a single occurrence word where $w \cap u = \{\varepsilon\}$ where $|\varepsilon| = 0$. Then $\mathcal{T}(u)$ acts on w so that

$$w \star \mathcal{T}(u) = xuyu'z$$

where

$$u' = \begin{cases} u & \mathcal{T} = \rho \\ u^R & \mathcal{T} = \tau \end{cases}$$

We call ρ the repeat insertion and τ the return insertion.

Example: Repeat and Return Insertion

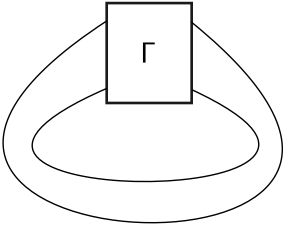
Let $w = 121323$ and let $u = 45$. Then we have that

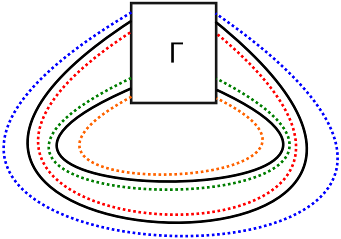
$$w \star \rho(u) = 1452134523$$
$$w \star \tau(u) = 1452135423$$

Graphical Representation of a Repeat Insertion

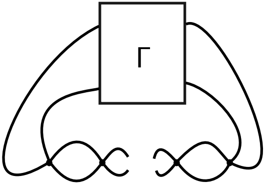
Let Γ_w be the assembly graph for some double occurrence word $w = xyz \in A_{DOW}$. Let u be a single occurrence word of length m. Then we can represent Γ_w and $\Gamma_{w\star\rho(u)}$ as

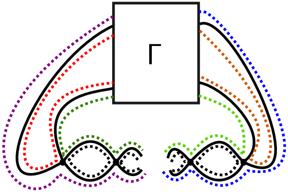
Γ:



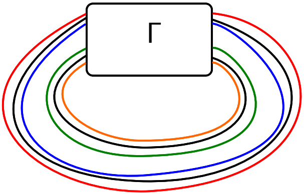


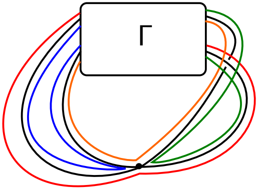
$\Gamma_{w\star\rho(u)}$:





External Connection Graphs





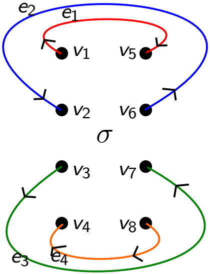


Fig 3: G

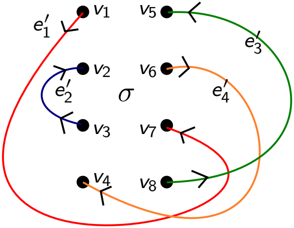


Fig 4: G'

External Connection Graphs

Definition: External Connection Graph

Definition 1. *Let* $w \in A_{DOW}$ *for some alphabet* A*. Let* $G = (V, E)$ *be a graph where* $V(G) = \{v_1, \ldots, v_8\}$, $\sigma' : \{v_1, v_3, v_6, v_8\} \rightarrow \{v_2, v_4, v_5, v_7\}$ *and* $\sigma : \{v_2, v_4, v_5, v_7\} \rightarrow \{v_1, v_3, v_6, v_8\}$ *are bijective functions, and* $E(G) = E_\sigma \dot\cup E_{ext}$ *where*

$$E_{ext} = \{(v_i, \sigma'(v_i)) | i = 1, 3, 6, 8\}$$
$$E_\sigma = \{(v_i, \sigma(v_i)) | i = 2, 4, 5, 7\}$$

We call G *the* **exterior connection graph** *of* $\Gamma_{w \star \rho(u,i,j)}$.

Theorem: Single Insertion of w

Theorem 1

Theorem 1. *Let* $A = \mathbb{N} \backslash \{1\}$ *be an alphabet and* $w \in A_{DOW}$*. Let* $\Gamma = \Gamma_w$ *be the assembly graph for* w *who's genus range is* $[a, b]$*. Then the assembly graph* $\Gamma' = \Gamma_{w \star \rho(1)}$ *for the double occurence word* $w \star \rho(1)$ *has a genus range of* $[a + \epsilon, b + \epsilon']$ *where* $\epsilon, \epsilon' \in \{-1, 0, 1, 2\}$

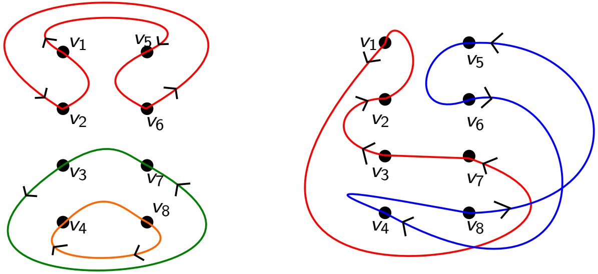
M_G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	$+1$	0	0	0	-1	0	0	0
v_2	0	-1	0	0	ζ_{25}	ζ_{26}	ζ_{27}	ζ_{28}
v_3	0	0	$+1$	0	0	-1	0	0
v_4	0	0	0	-1	ζ_{45}	ζ_{46}	ζ_{47}	ζ_{48}
v_5	-1	0	0	0	ζ_{55}	ζ_{56}	ζ_{57}	ζ_{58}
v_6	0	$+1$	0	0	0	0	-1	0
v_7	0	0	-1	0	ζ_{75}	ζ_{76}	ζ_{77}	ζ_{78}
v_8	0	0	0	$+1$	0	0	0	-1

$$\downarrow$$

M'_G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	$+1$	0	0	0	-1	0	0	0
v_2	0	-1	0	0	ζ_{25}	ζ_{26}	ζ_{27}	ζ_{28}
v_3	0	$+1$	0	0	0	-1	0	0
v_4	0	0	-1	0	ζ_{45}	ζ_{46}	ζ_{47}	ζ_{48}
v_5	0	0	0	-1	ζ_{55}	ζ_{56}	ζ_{57}	ζ_{58}
v_6	0	0	$+1$	0	0	0	-1	0
v_7	-1	0	0	0	ζ_{75}	ζ_{76}	ζ_{77}	ζ_{78}
v_8	0	0	0	$+1$	0	0	0	-1

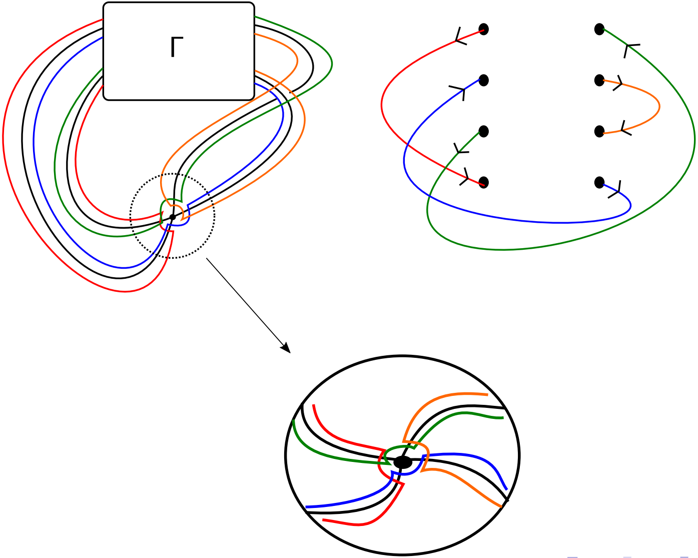
Example

Let $\sigma((v_1, v_3, v_6, v_8)) = (v_2, v7, v_5, v_4)$. Then the external connection graphs for Γ_w and $\Gamma_{w\star xyz}$ are



Permutation Type	$\lvert b(\Gamma)\rvert$	$\lvert b(\Gamma')\rvert$	$\lvert b(\Gamma')\rvert - \lvert b(\Gamma)\rvert$
(v_2, v_4, v_5, v_7)	2	1	-1
(v_2, v_4, v_7, v_5)	1	2	1
(v_2, v_5, v_4, v_7)	1	2	1
(v_2, v_5, v_7, v_4)	2	1	-1
(v_2, v_7, v_4, v_5)	2	3	1
(v_2, v_7, v_5, v_4)	3	2	-1
(v_4, v_2, v_5, v_7)	1	2	1
(v_4, v_2, v_7, v_5)	2	3	1
(v_4, v_5, v_2, v_7)	2	1	-1
(v_4, v_5, v_7, v_2)	1	2	1
(v_4, v_7, v_2, v_5)	3	2	-1
(v_4, v_7, v_5, v_2)	2	1	-1
(v_5, v_2, v_4, v_7)	2	3	1
(v_5, v_2, v_7, v_4)	3	2	-1
(v_5, v_4, v_2, v_7)	3	2	-1
(v_5, v_4, v_7, v_2)	2	1	-1
(v_5, v_7, v_2, v_4)	4	1	-3
(v_5, v_7, v_4, v_2)	3	2	-1
(v_7, v_2, v_4, v_5)	1	4	3
(v_7, v_2, v_5, v_4)	2	3	1
(v_7, v_4, v_2, v_5)	2	3	1
(v_7, v_4, v_5, v_2)	1	2	1
(v_7, v_5, v_2, v_4)	3	2	-1
(v_7, v_5, v_4, v_2)	2	3	1

We then consider what occurs when we change the boundary at the inserted vertex. G'' in this context is the external connection graph for Γ with a boundary connection change at the inserted vertex.



Permutation Type	$\lvert b(\Gamma)\rvert$	$\lvert b'(\Gamma')\rvert$	$\lvert b'(\Gamma')\rvert - \lvert b(\Gamma)\rvert$
(v_2, v_4, v_5, v_7)	2	1	-1
(v_2, v_4, v_7, v_5)	1	2	1
(v_2, v_5, v_4, v_7)	1	2	1
(v_2, v_5, v_7, v_4)	2	3	1
(v_2, v_7, v_4, v_5)	2	1	-1
(v_2, v_7, v_5, v_4)	3	2	-1
(v_4, v_2, v_5, v_7)	1	2	1
(v_4, v_2, v_7, v_5)	2	3	1
(v_4, v_5, v_2, v_7)	2	3	1
(v_4, v_5, v_7, v_2)	1	4	3
(v_4, v_7, v_2, v_5)	3	2	-1
(v_4, v_7, v_5, v_2)	2	3	1
(v_5, v_2, v_4, v_7)	2	1	-1
(v_5, v_2, v_7, v_4)	3	2	-1
(v_5, v_4, v_2, v_7)	3	2	-1
(v_5, v_4, v_7, v_2)	2	3	1
(v_5, v_7, v_2, v_4)	4	1	-3
(v_5, v_7, v_4, v_2)	3	2	-1
(v_7, v_2, v_4, v_5)	1	2	1
(v_7, v_2, v_5, v_4)	2	1	-1
(v_7, v_4, v_2, v_5)	2	1	-1
(v_7, v_4, v_5, v_2)	1	2	1
(v_7, v_5, v_2, v_4)	3	2	-1
(v_7, v_5, v_4, v_2)	2	3	1

$$g(\Gamma') = \frac{1}{2}(2 + (n+1) - b(\Gamma')) = \frac{1}{2}(2 + n - b(\Gamma')) + \frac{1}{2}(1 - \epsilon)$$

where $\epsilon = -3, -1, 1, 3$ so that

$$g(\Gamma') = g(\Gamma) + \delta$$

where $\delta = -1, 0, 1, 2$.

Theorem 2

Theorem 2. *Let A be an alphabet and $u, v, w \in A$ so that $uvw \in A_{DOW}$. Let Γ be the assembly graph for uvw who's genus range is $[a, b]$. Then the assembly graph Γ' for the double occurence word $u12v12w$ has a genus range of $[a + \epsilon, b + \epsilon']$ where $\epsilon, \epsilon' \in \{0, 1, 2\}$*

Theorem 3

Theorem 3 **L.***et A be an alphabet $w \in A_{DOW}$. Let $\Gamma = \Gamma_w$ be the assembly graph for w who's genus range is $[a, b]$. Let $v_{odd}, v_{even}, u \in A_{SOW}$ where $|v_{odd}| = 3, |v_{even}| = 2$, and $|u| = n$.*

$$g(\Gamma_{w \star \mathcal{T}(u,i,j)}) = \begin{cases} g(\Gamma_{w \star \mathcal{T}(v_{odd},i,j)}) & \text{If } n \equiv \text{ odd} \\ g(\Gamma_{w \star \mathcal{T}(v_{even},i,j)} & \text{If } n \equiv \text{ even} \end{cases}$$

Example

Consider the *DOW*

$$12145673234567 = 121323 \star \rho(4567)$$

Example

Consider the *DOW*

$$12145673234567 = 121323 \star \rho(4567)$$

We have that $g(121323 \star \rho(4567)) = g(121323 \star \rho(45)) = [2, 3]$

Conclusion

- Construction of a lower bound
- Classes of assembly graphs after subgraph deletion

References

- D. Buck, E. Dolzhenko, N. Jonoska, M. Saito, K. Valencia. Genus Ranges of 4-Regular Rigid Vertex Graphs. *The Electronic Journal of Combinatorics*. pg (1-10,15)
- N. Jonoska, L. Nabergall, M. Saito. Patterns and Distances in Words Related to DNA Rearrangement. *Fundamenta Informaticae*. pg(1003 - 1007)
- D. Cruz, M. Ferrari, N. Jonoska, L. Nabergall, M. Saito. Transformations on Double Occurrence Words Motivated by DNA Rearrangement. arXiv:1811.11739 . pg(5,6)

Thank you!