Calculating the Braid Index of Chord Diagrams

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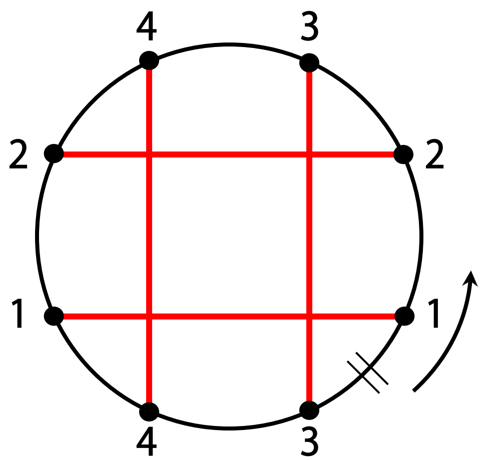
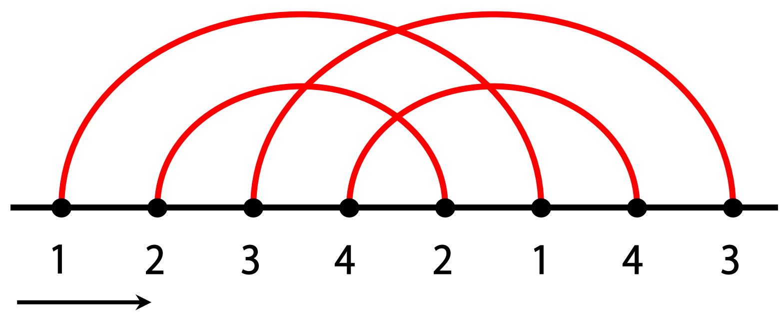
March 6, 2013

Forty-Fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing

Chord Diagrams



Chord Diagram (CD)



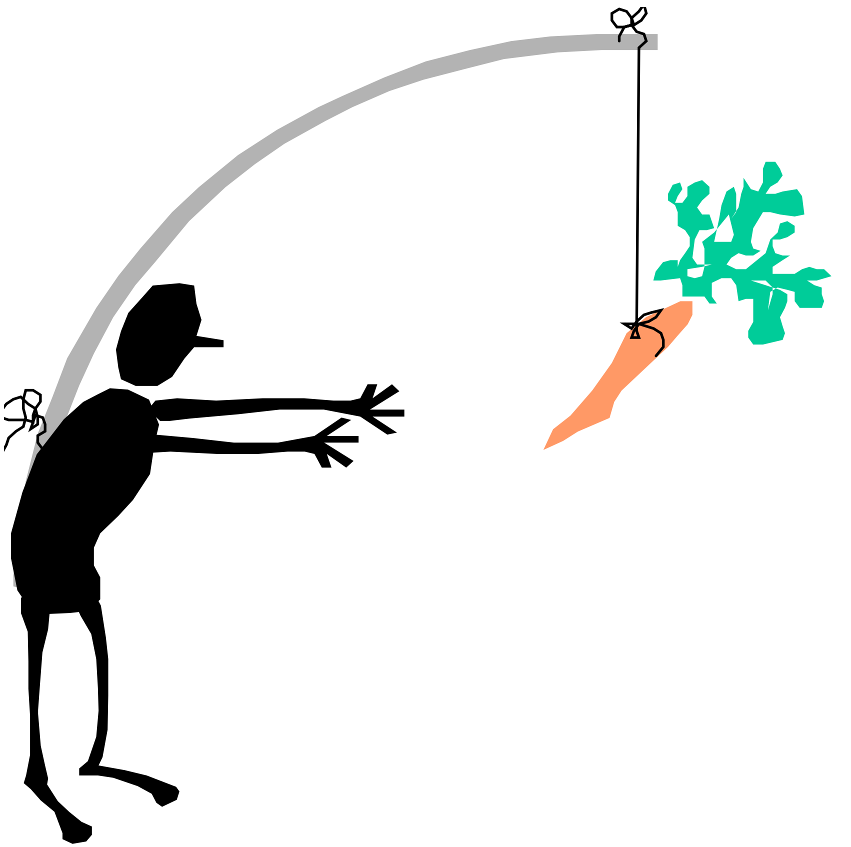
Linear Chord Diagram (LCD)

Chord diagram :

- A Backbone ( )
- Chords ( )

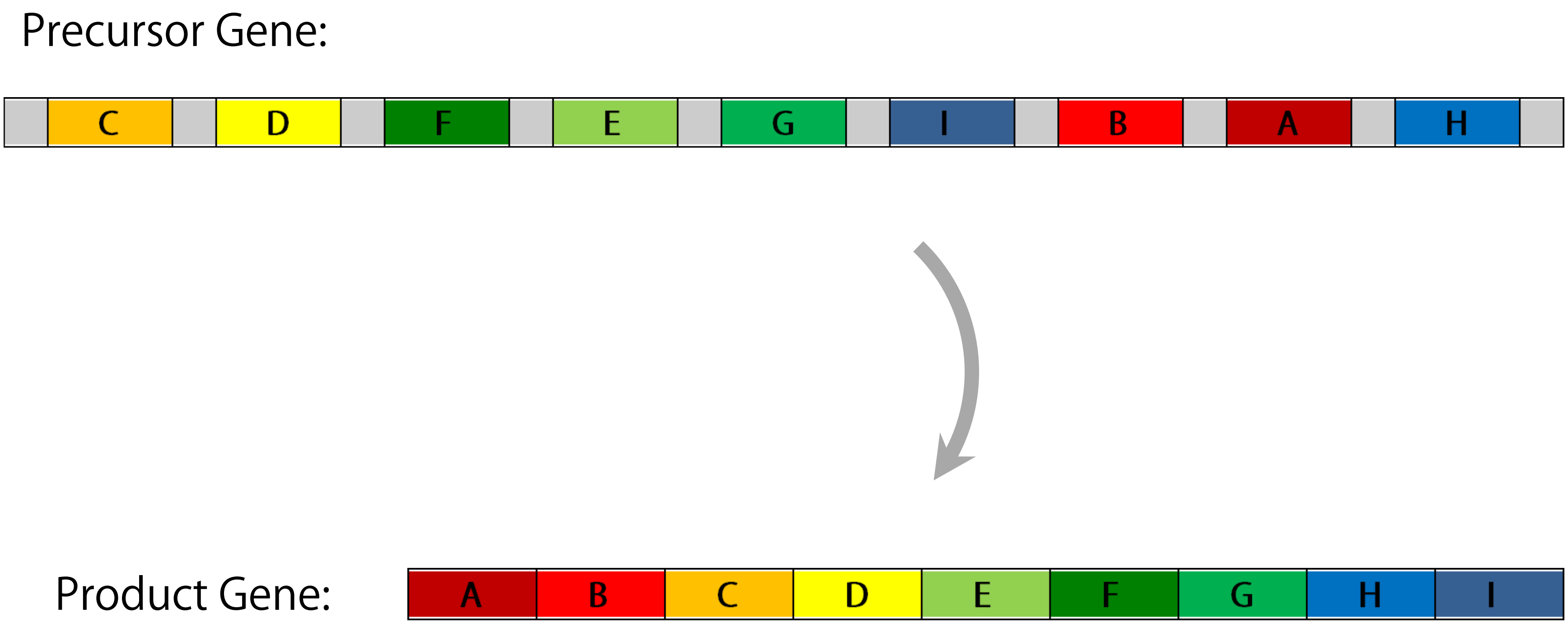
Generalizations :

- Base Points
- Orientations
- Multiple Backbones



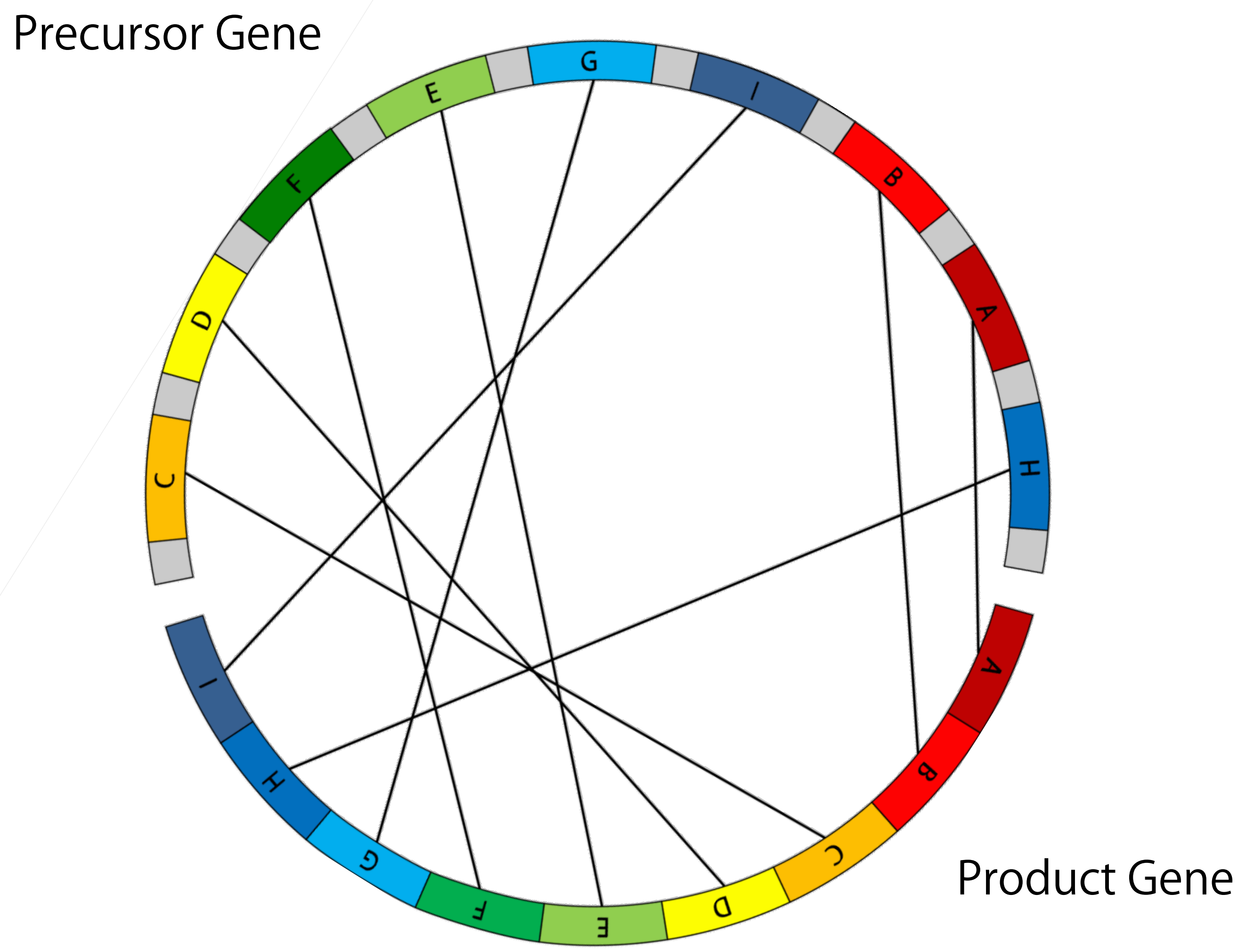
Biological Motivation

DNA Recombination



Actin I (*Oxytricha Nova*)

DNA Recombination



Oxytricha Tryallifax

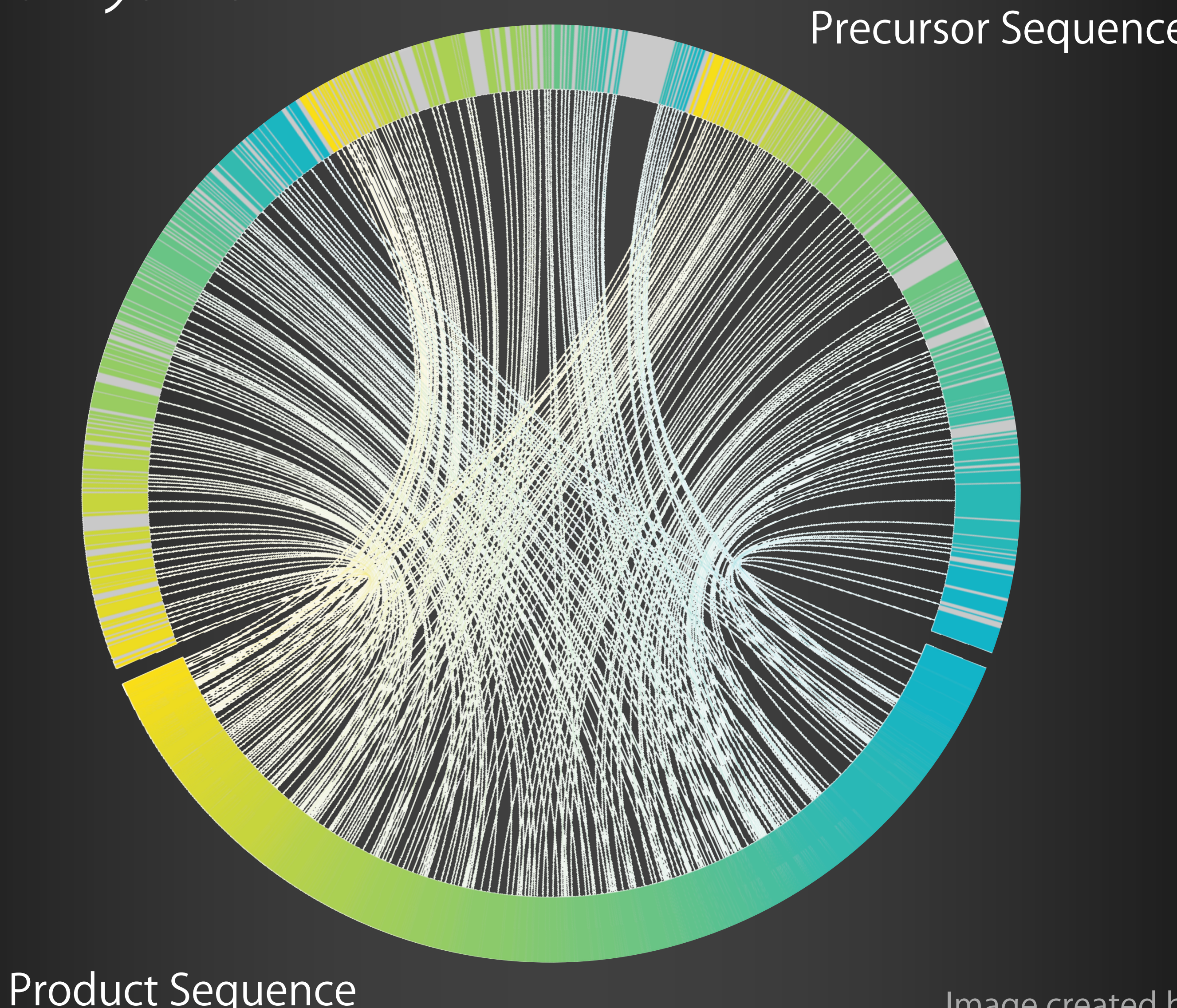
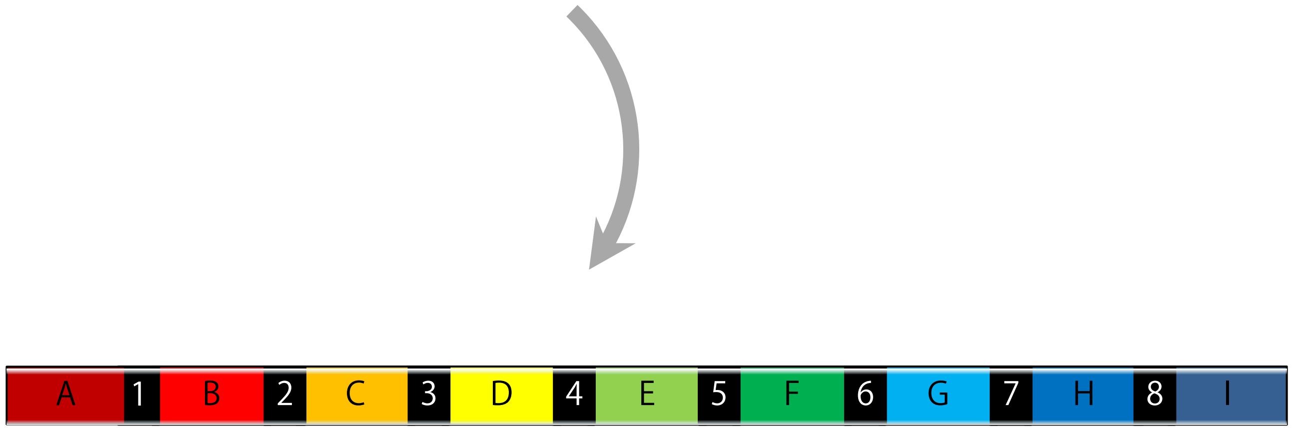


Image created by STAGR
courtesy of Egor Dolzhenko

Assembly Pathways

Precursor Gene:

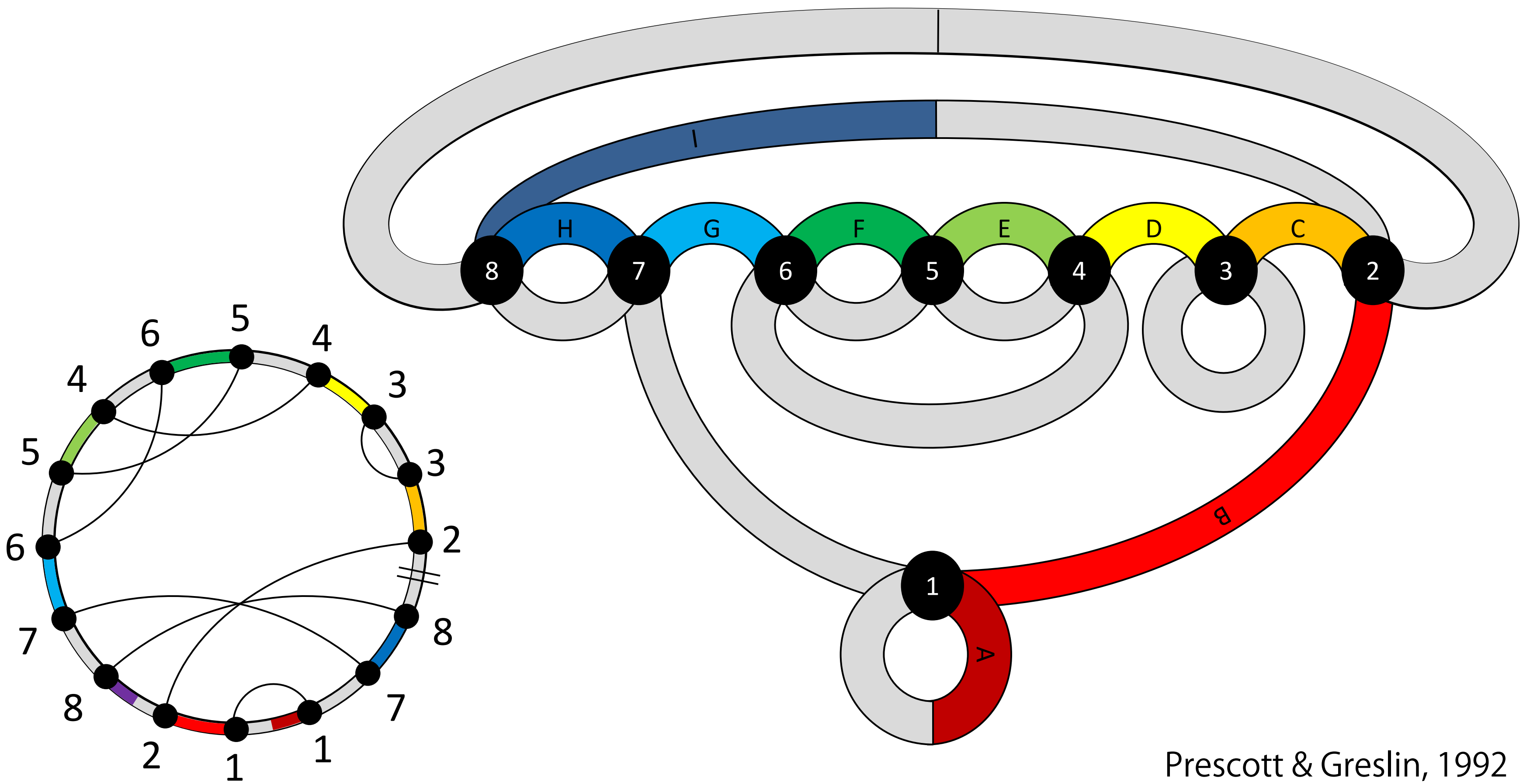
Product Gene:



Actin I (*Oxytricha Nova*)

Assembly Pathways

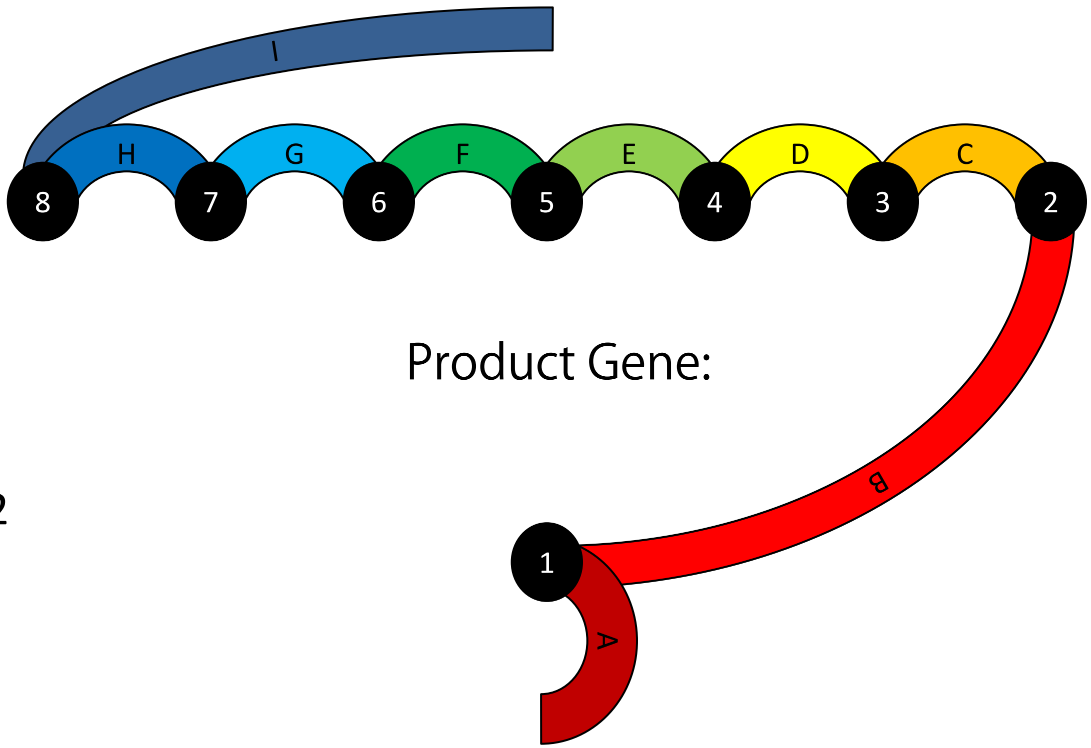
Precursor Gene:



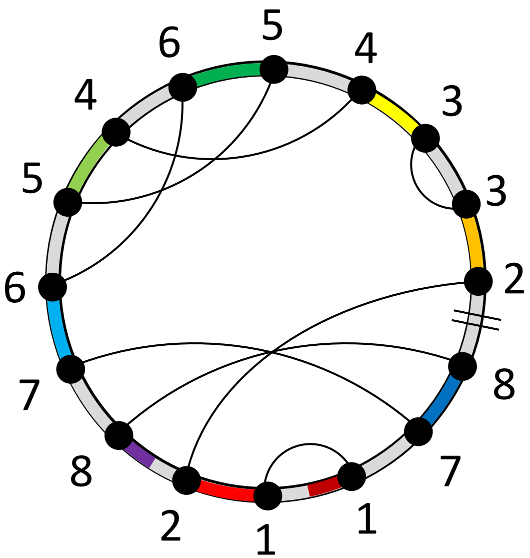
Prescott & Greslin, 1992

Assembly Pathways

Precursor Gene:



Product Gene:



Prescott & Greslin, 1992

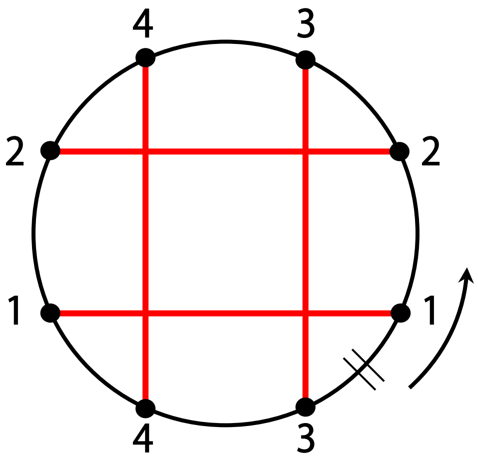
Biological Motivation

1. Chord diagrams arise naturally in genetics

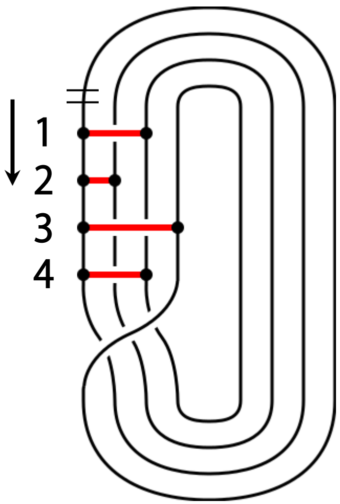
2. Complexity measures of chord diagrams *may* be useful for determining evolutionary relationships

Braid Index for CDs

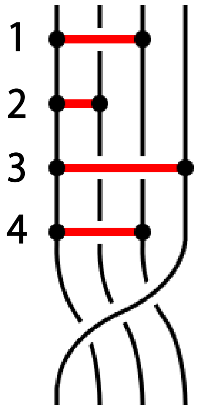
Chorded Braid Diagrams



Chord Diagram

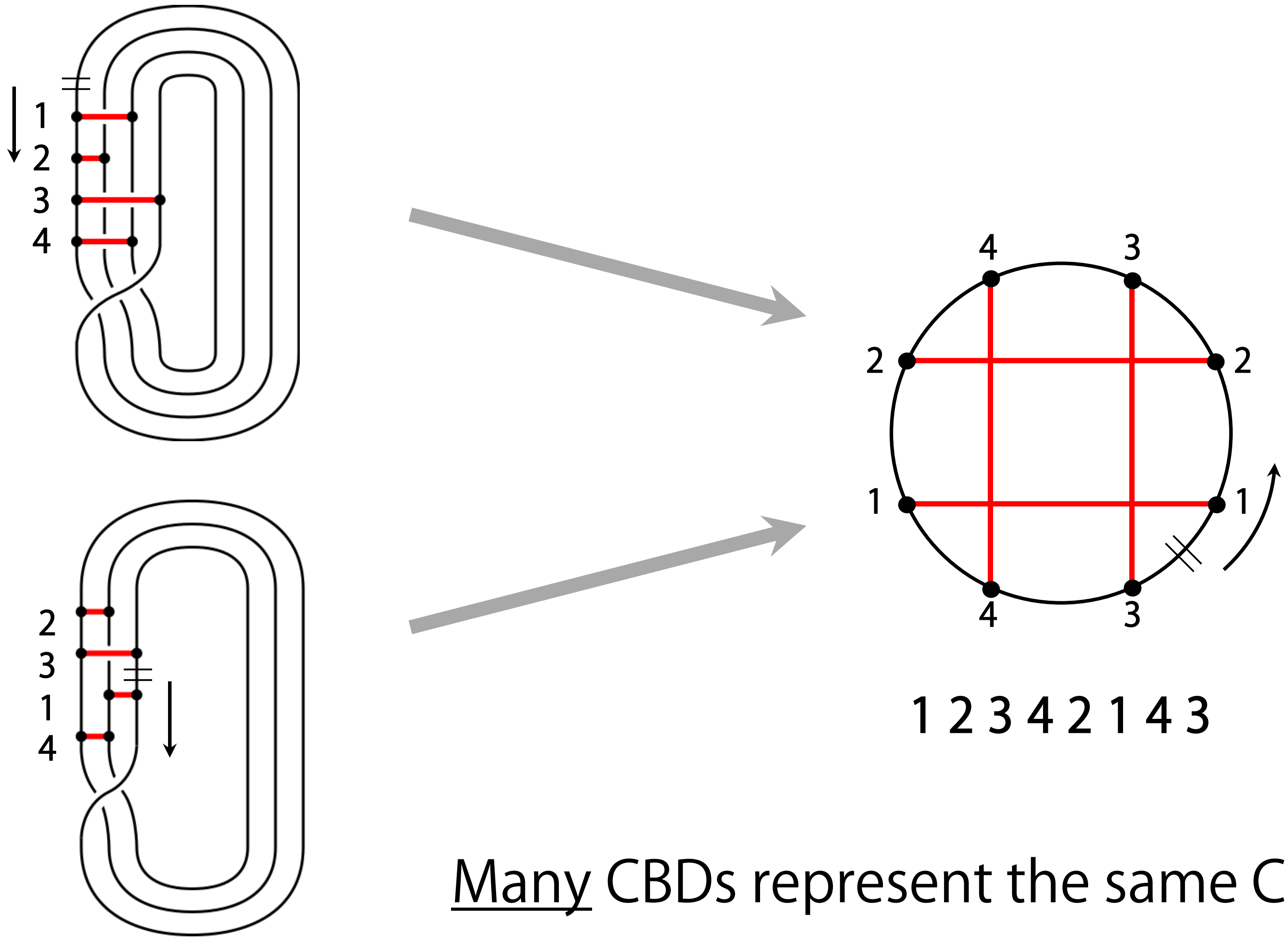


Closure of CBD

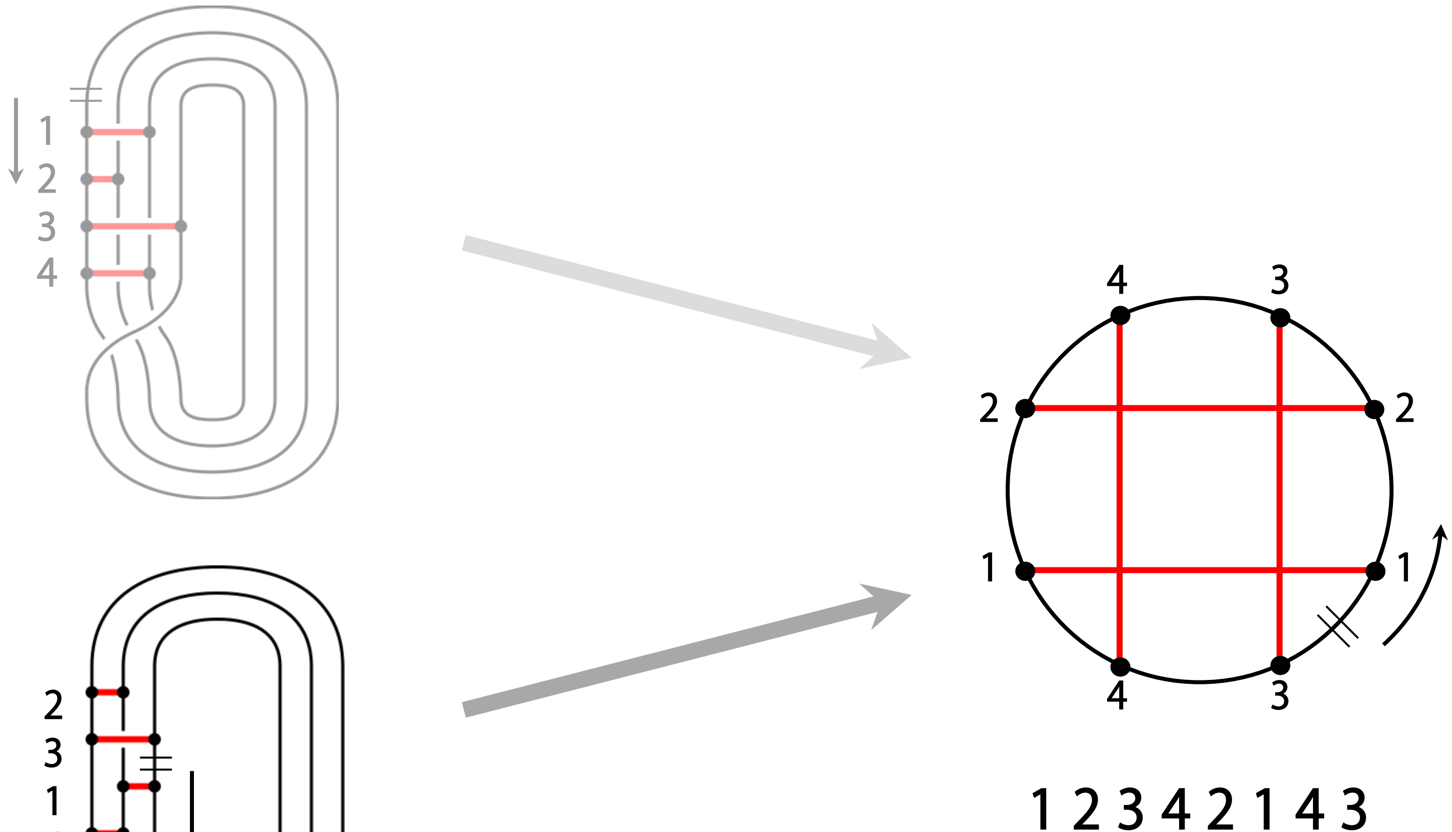


Chorded Braid Diagram

Each chord diagram has an associated **chorded braid diagram (CBD)** which consists of a braid whose strings are connected by non-intersecting chords.

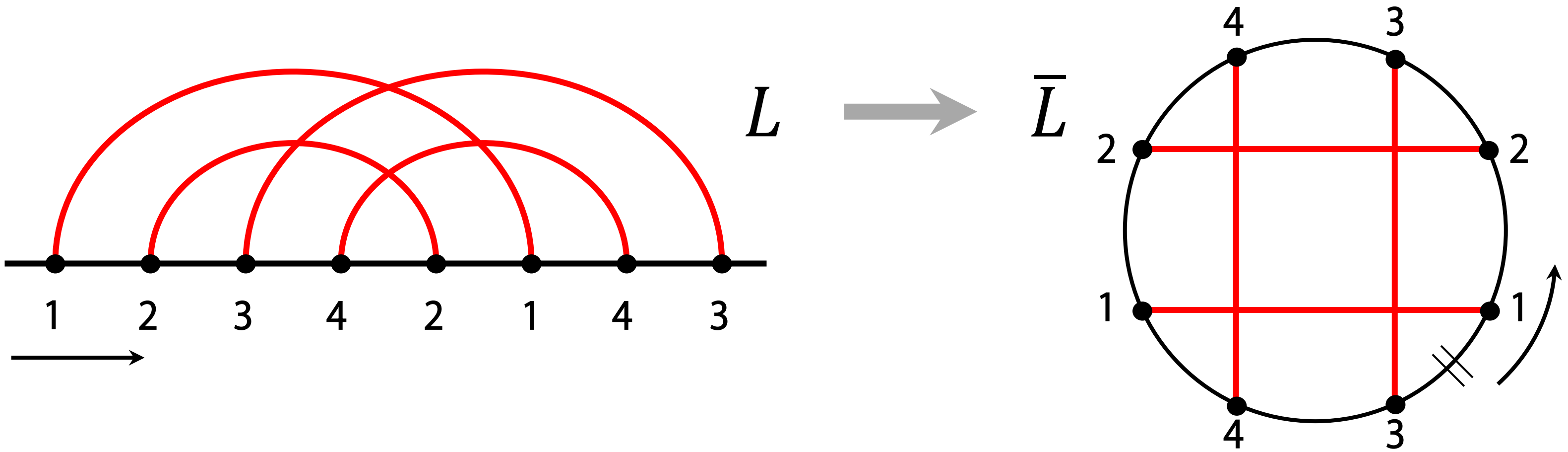


<u>Many</u> CBDs represent the same CD

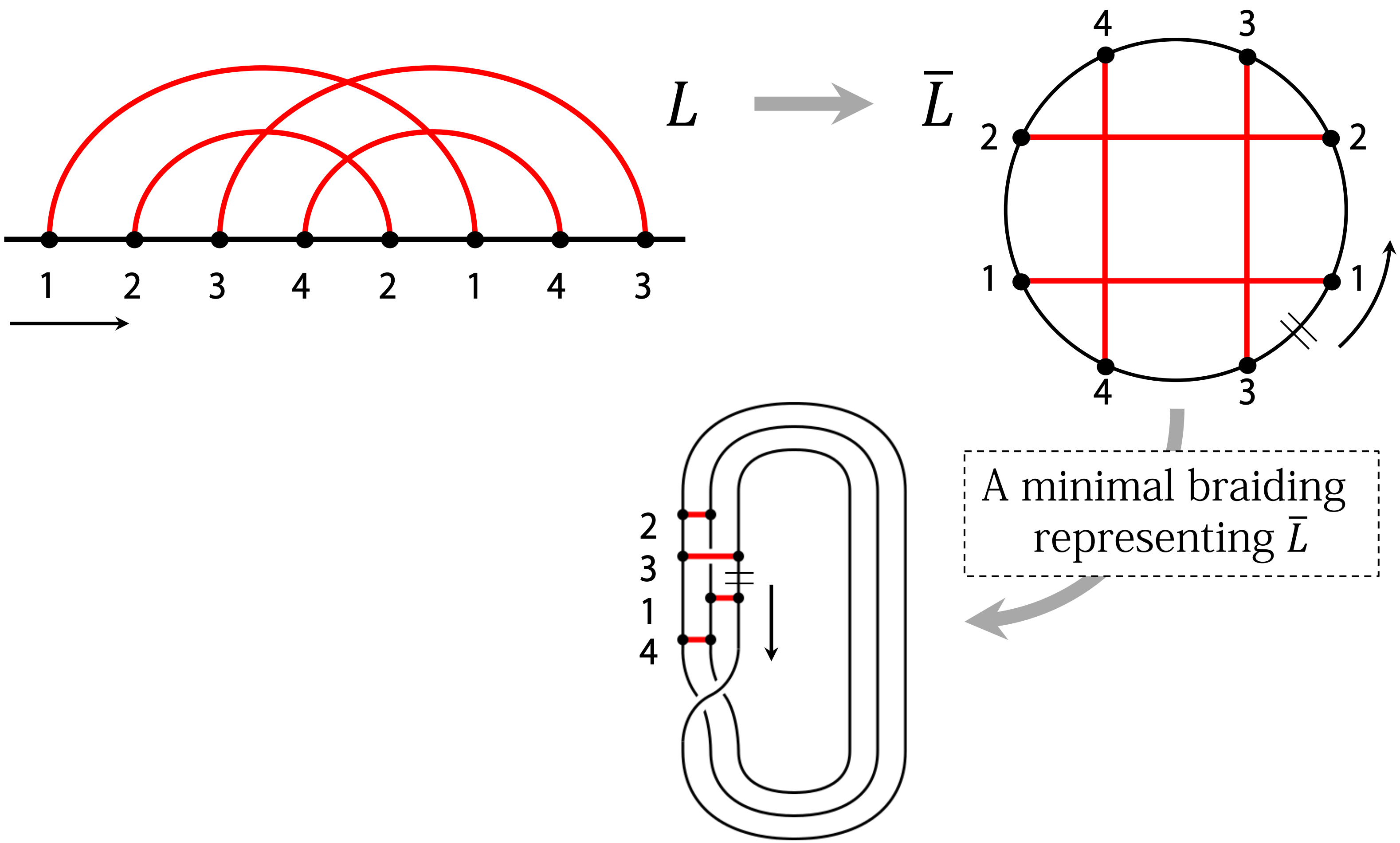


The **braid index** of a CD is the *least* number of strands used in any braid representative

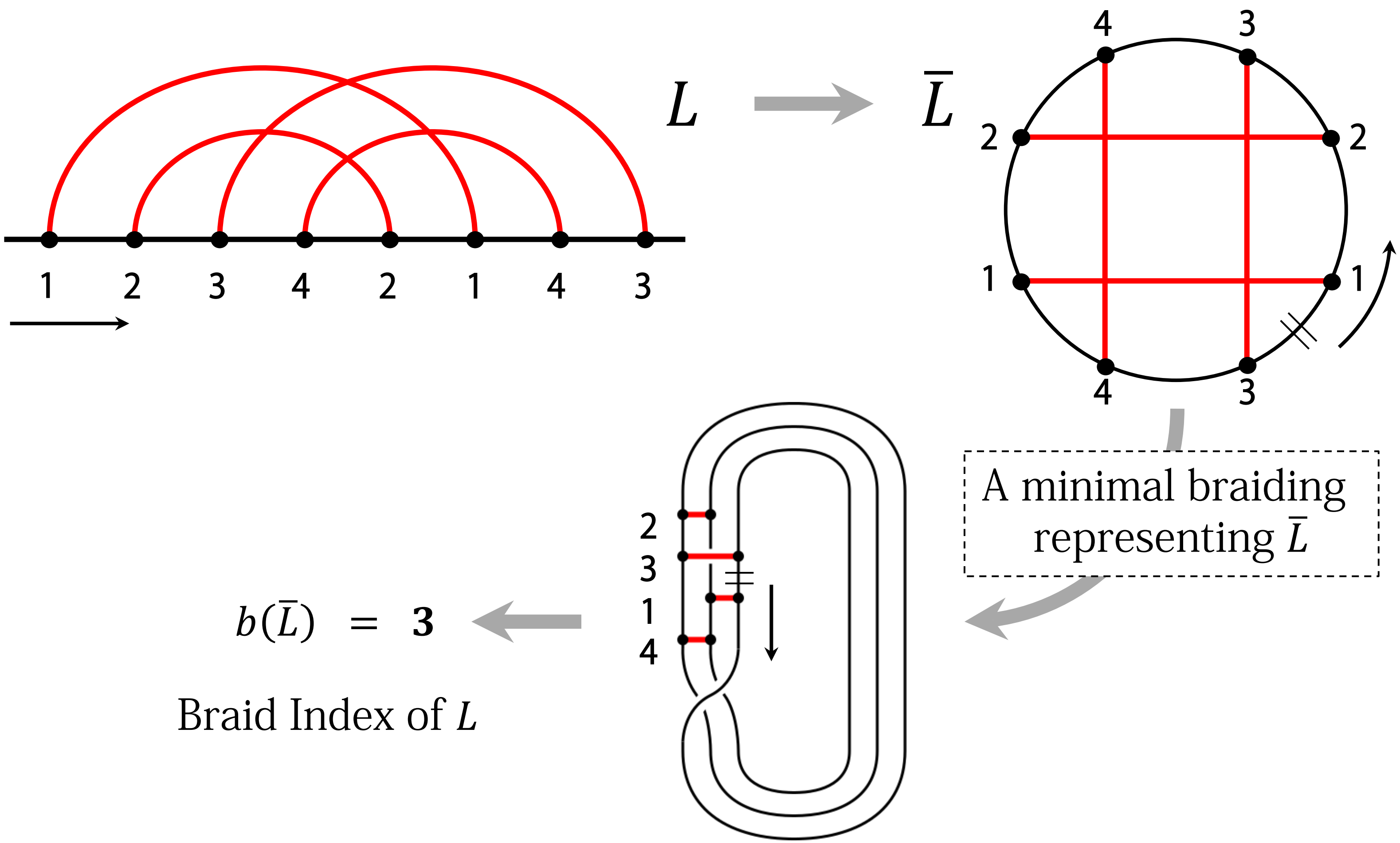
Braid Index of a LCD



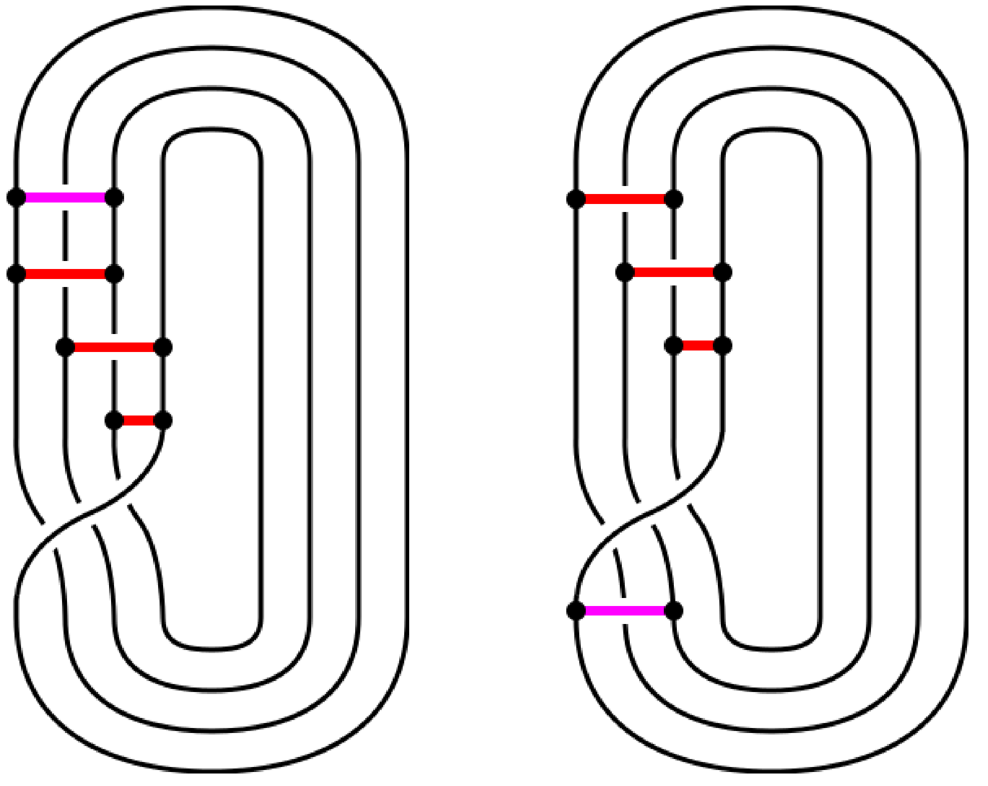
Braid Index of a LCD



Braid Index of a LCD



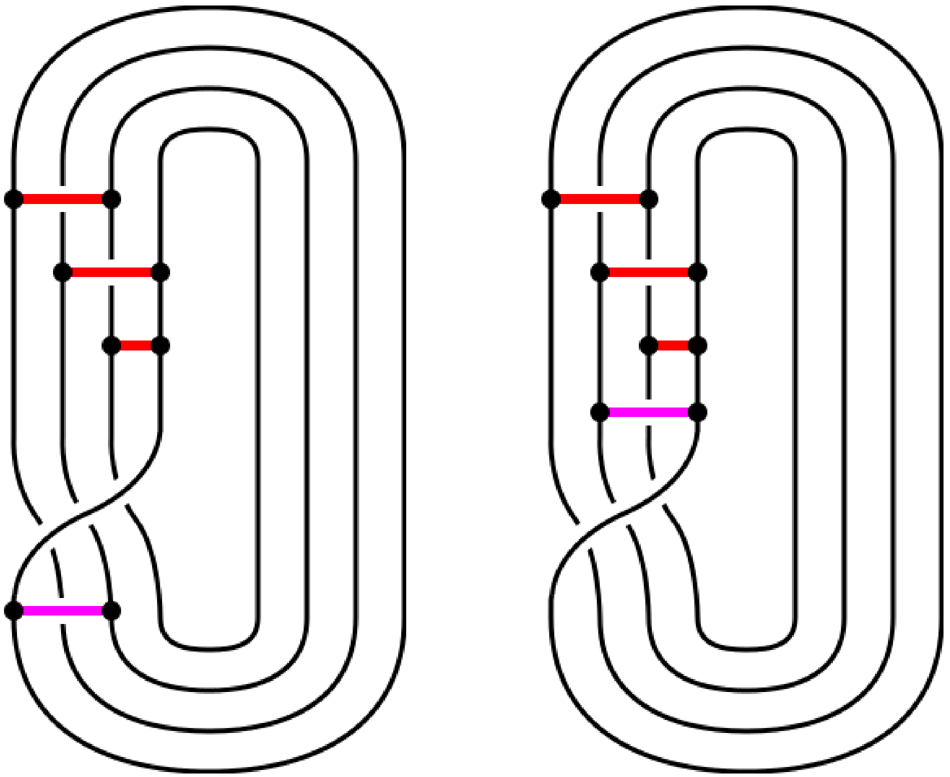
Operations Preserving CDs



Cyclic Permutation

Birman & Trapp, 1998

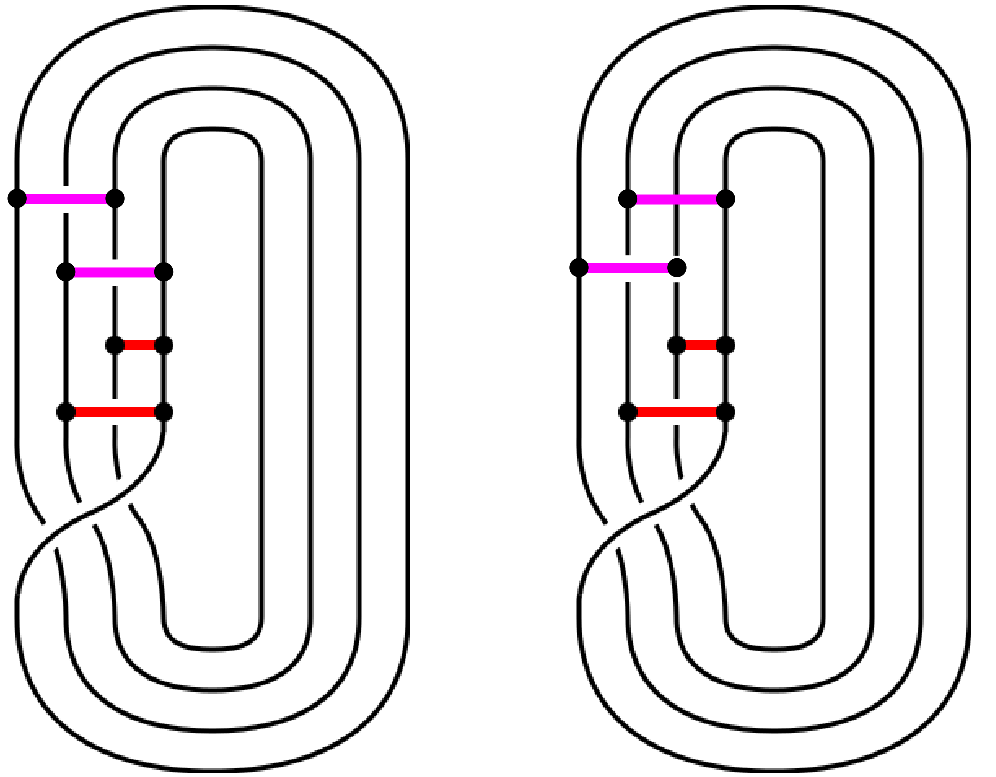
Operations Preserving CDs



Cyclic Permutation

Birman & Trapp, 1998

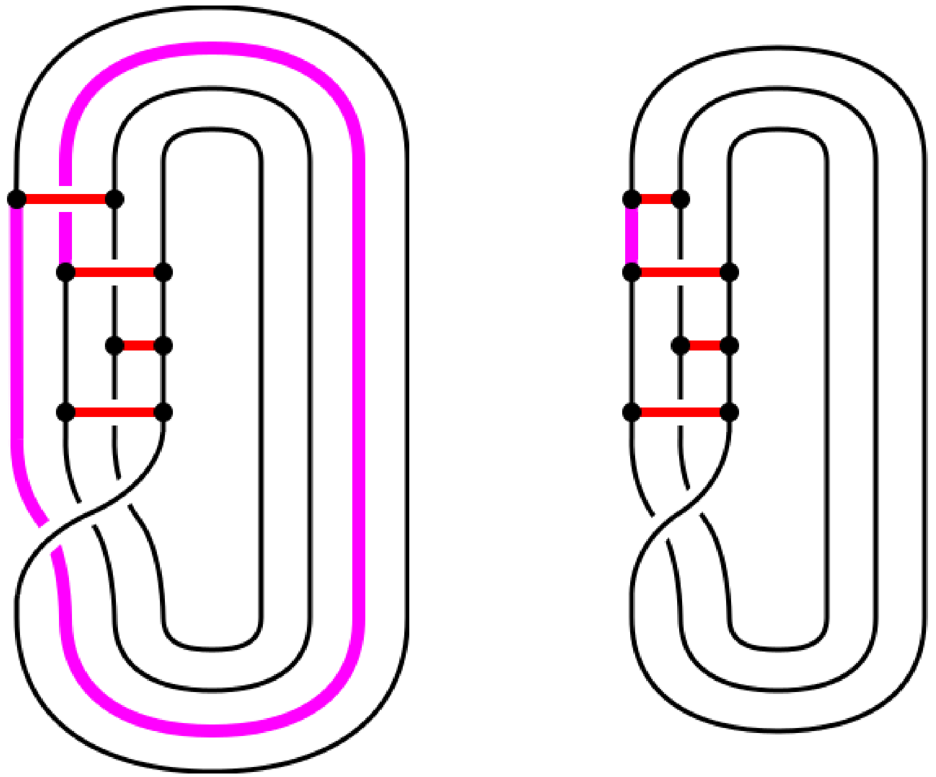
Operations Preserving CDs



Commutation

Birman & Trapp, 1998

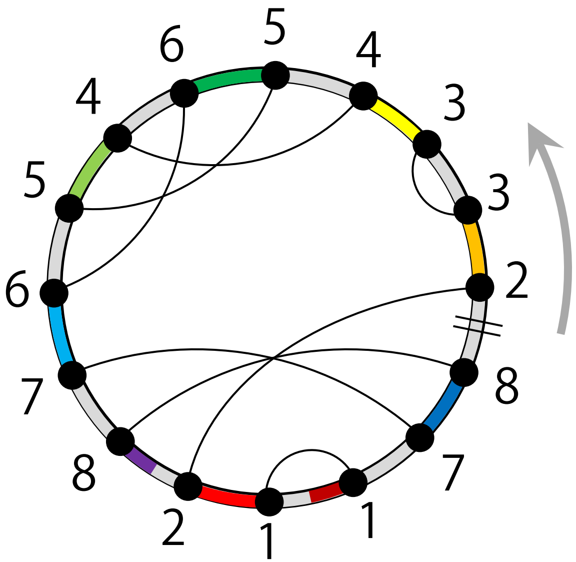
Operations Preserving CDs



Contracting/Expanding Trivial Loops

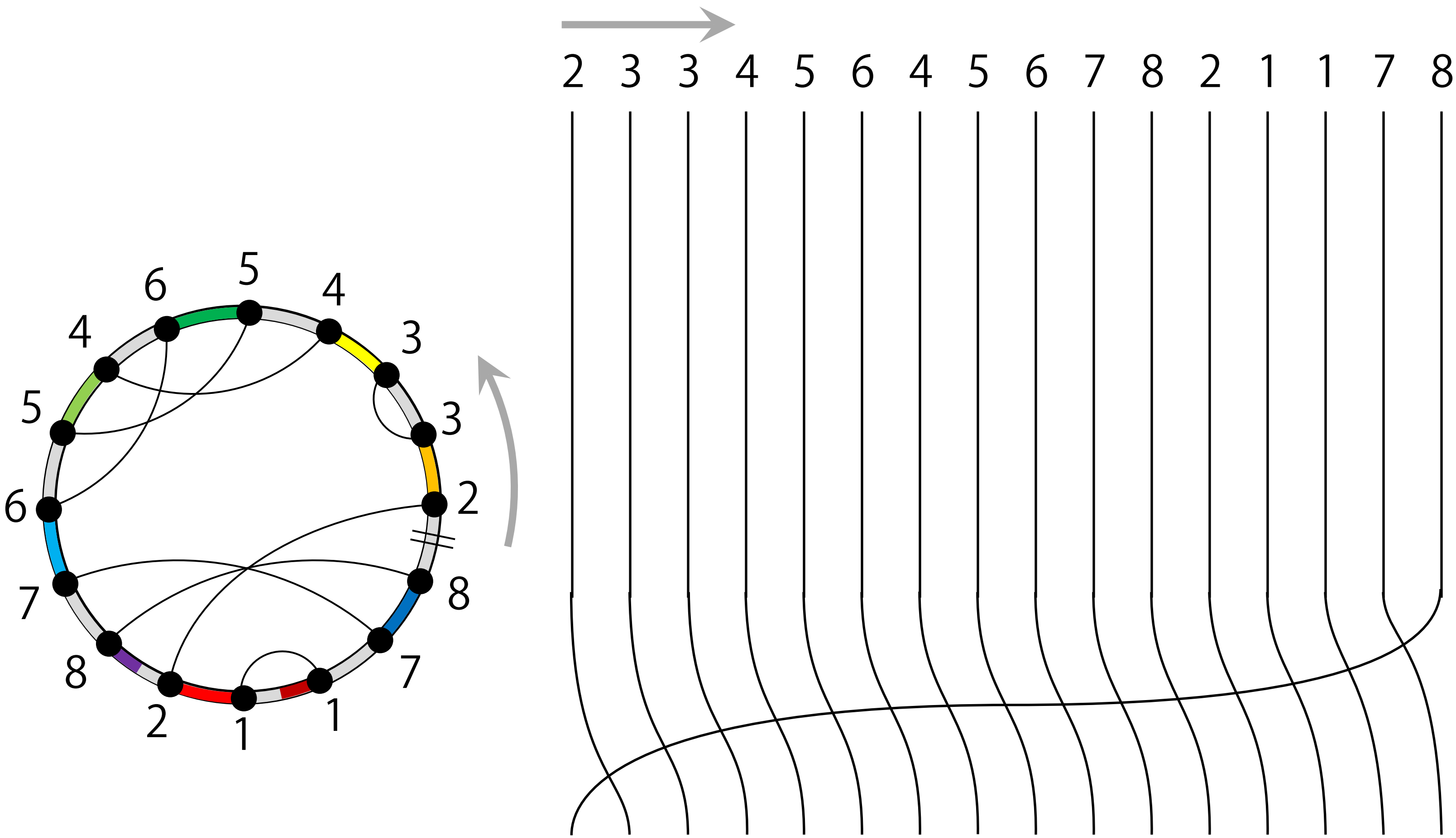
Birman & Trapp, 1998

How to Find a CBD from a CD



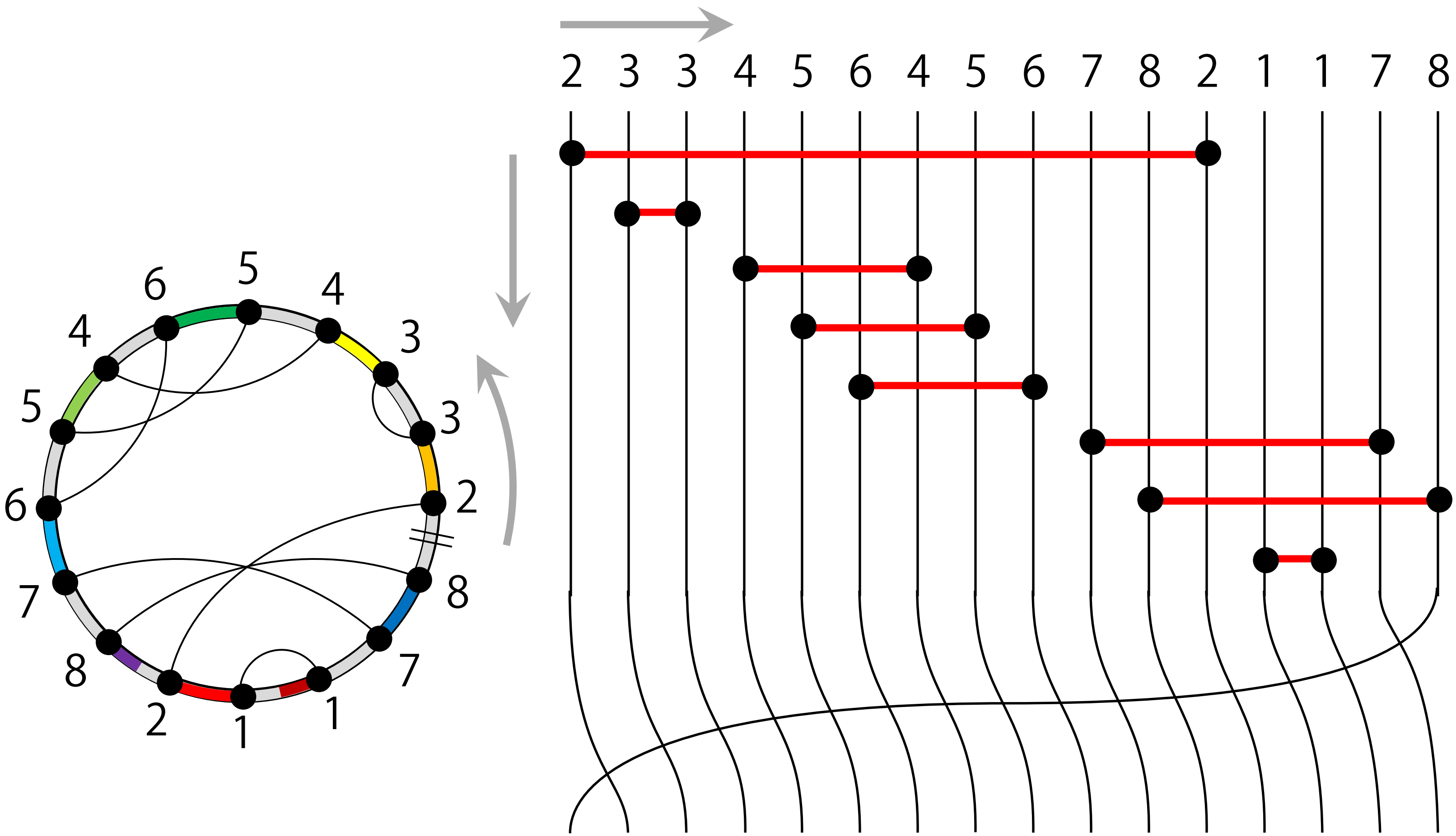
Birman & Trapp, 1998

How to Find a CBD from a CD



Birman & Trapp, 1998

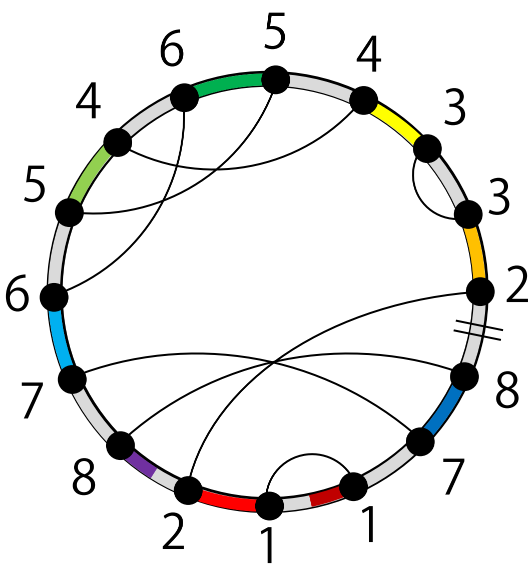
How to Find a CBD from a CD

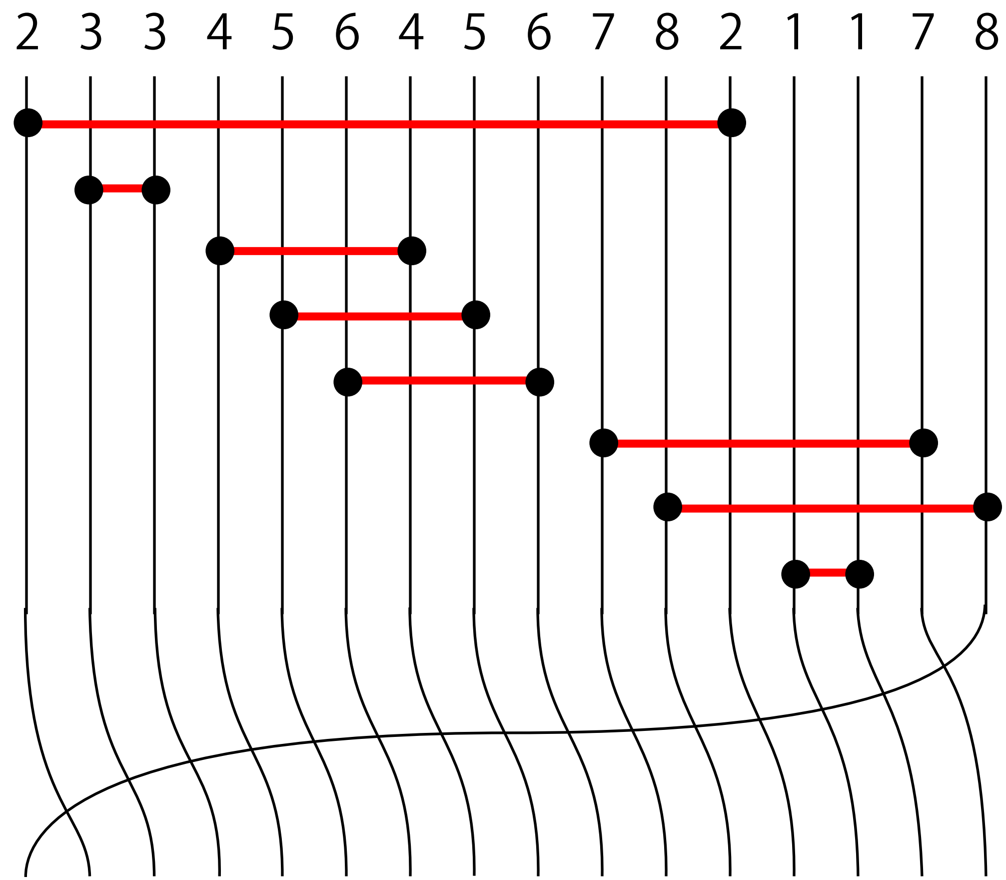


Canonical Braiding

Birman & Trapp, 1998

How to Find a CBD from a CD





All Chords Commute!!!

Birman & Trapp, 1998

$(n-1)!$ Possibilities

Calculating Braid Index

1. Fix base point.
2. Generate the canonical braiding.
3. Transverse the braid, removing all trivial loops.
4. Record the number of strings in the resulting braiding.
5. Loop back to Step 2 until *all possible ways* to commute the chords in canonical braiding have been exhausted.

Pseudocode

Mathematica

Function CanonicalBraid(N)

Input: A chord diagram D_n with name N

Output: $W = A(i_1, j_1) \dots A(i_n, j_n)$

$W \leftarrow \emptyset$;
for $k \leftarrow 1$ **to** n **do**
  $W \leftarrow WA($position$(N, k, 1)$, position$(N, k, 2))$;
  // where position(N, k, j) returns the position of the j^{th} letter k in N
end
return W;

Function DecreasingStablilization(W)

Input: $W = A(i_1, j_1) \dots A(i_n, j_n)$ corresponding to a chord diagram D_n with name N

Output: W which closes to form a braid containing no trivial loops.

for $k \leftarrow 1$ **to** n **do**
  if $i_k - 1 \notin \{i_t, j_t \mid k \leq t \leq n\}$ *and* $i_k \notin \{i_t, j_t \mid 1 \leq t \leq k-1\}$ **then**
    $W \leftarrow$ Decrease(W, i_k) ;
    // where Decrease(W, c) decreases $i_t, j_t \leq c$ by 1 for all generators of W
  end
  if $j_k - 1 \notin \{i_t, j_t \mid k \leq t \leq n\}$ *and* $j_k \notin \{i_t, j_t \mid 1 \leq t \leq k-1\}$ **then**
    $W \leftarrow$ Decrease(W, j_k) ;
  end
end
$W \leftarrow$ Increase(W) ;
// where Increase(W) increases i_t, j_t by 1 for all generators of W
return W;

Algorithm 1: Computing Braid Index of a Chord Diagram

Input: A chord diagram D_n with name N

Output: $br(D_n)$

$m \leftarrow n + 1$;
$W^* \leftarrow$ CanonicalBraid(N) ;
for $k \leftarrow 1$ **to** $(n-1)!$ **do**
  $W_k \leftarrow \sigma^{(1)}_{n,k}(W^*)$; // k^{th} permutation of S_n fixing 1
  $A(i_1, j_1) \dots A(i_n, j_n) \leftarrow$ DecreasingStablilization(W_k) ;
  $m \leftarrow \min\{m, \max\{i_t, j_t \mid 1 \leq t \leq n\}\}$;
end
return m ;

CD Braid Indices

Braid Index

Chords	2	3	4	5	6	7	8
1	1						
2	1	1					
3	1	2	2				
4	1	6	7	3			
5	1	12	38	22	6		
6	1	29	198	235	79	12	
7	1	65	1,107	2,390	1,409	284	27

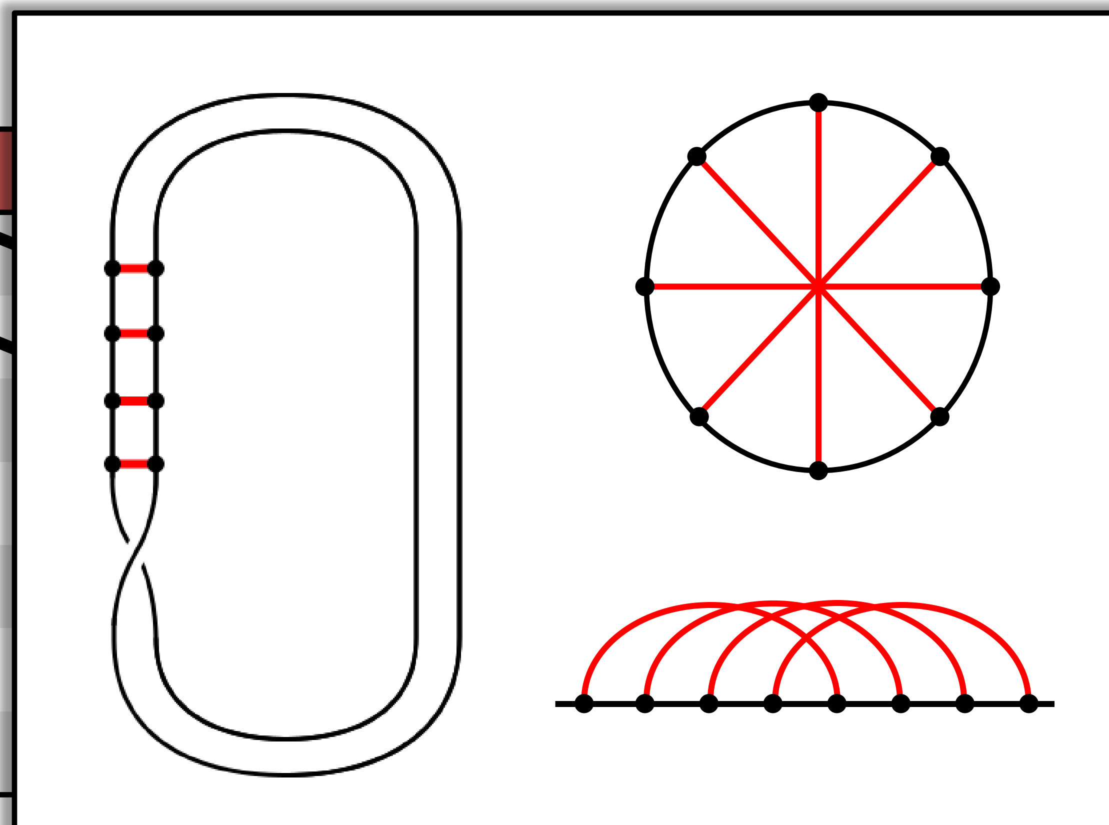
CD Braid Indices

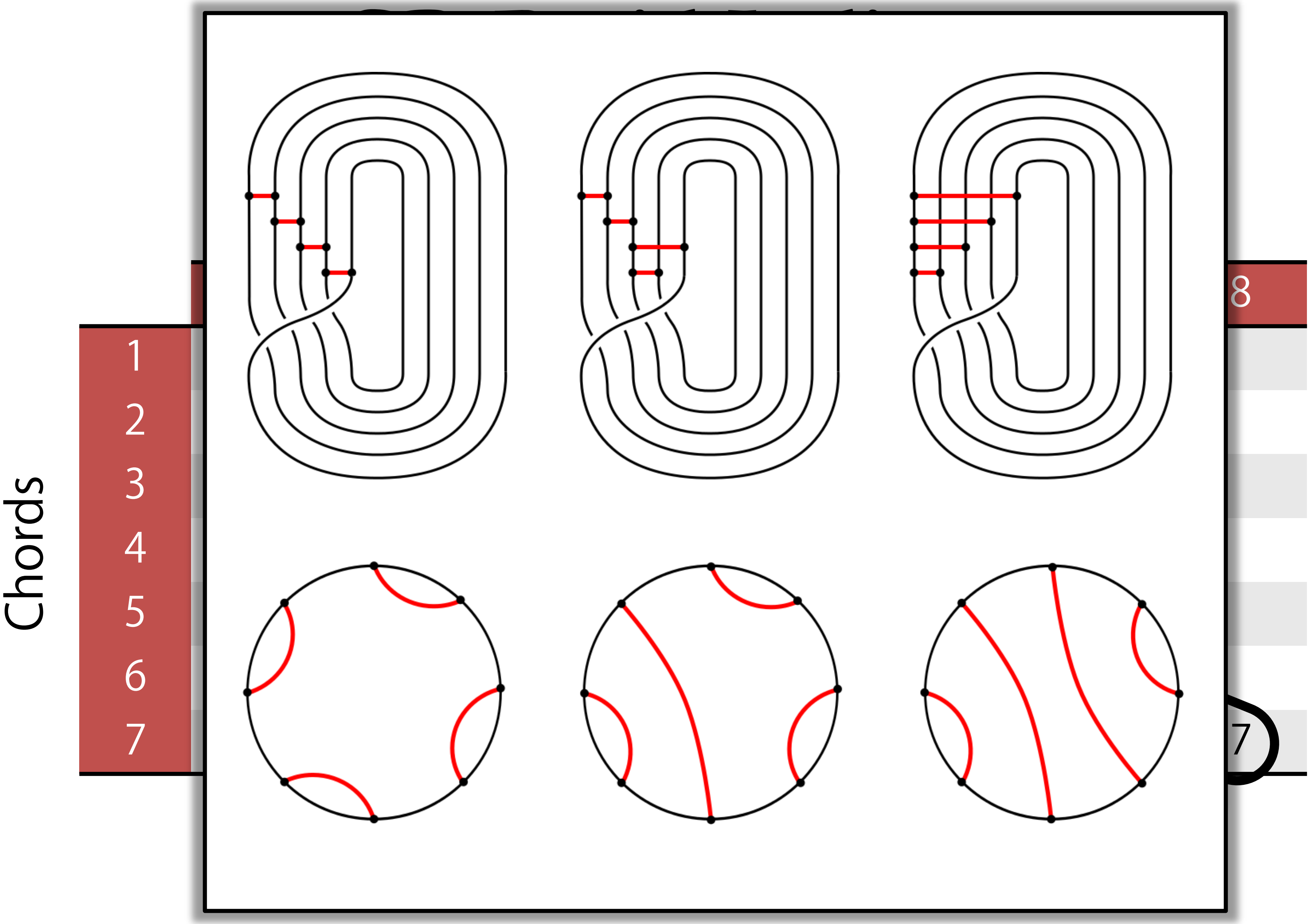
Braid Index

Chords	2	3	4	5	6	7	8
1	1						
2	1	1					
3	1	2	2				
4	1	6	7	3			
5	1	12	38	22	6		
6	1	29	198	235	79	12	
7	1	65	1,107	2,390	1,409	284	27

CD Braid Indices

Chords	2	8
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	1	27




Chords
1
2
3
4
5
6
7
8
7

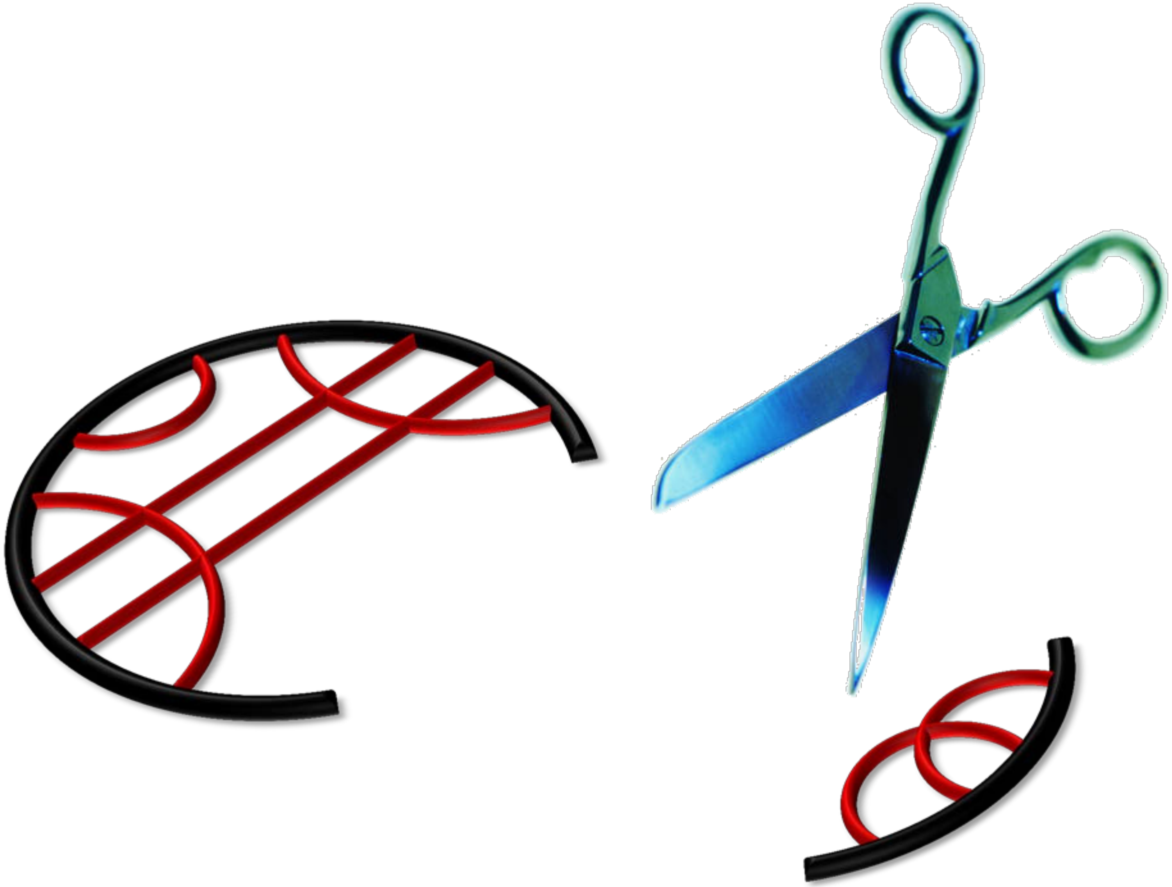
LCD Braid Indices

Braid Index

Chords	2	3	4	5	6	7	8
1	1						
2	1	1					
3	1	9	5				
4	1	34	56	14			
5	1	122	480	300	42		
6	1	414	3,743	4,620	1,485	132	
7	1	1,339	27,554	62,769	36,036	7,007	429

$(2\cdot\mathbf{7}-1)!! = 135{,}135$ Total LCDs

$(\mathbf{7}-1)! = 720$ Canonical Braidings / LCD

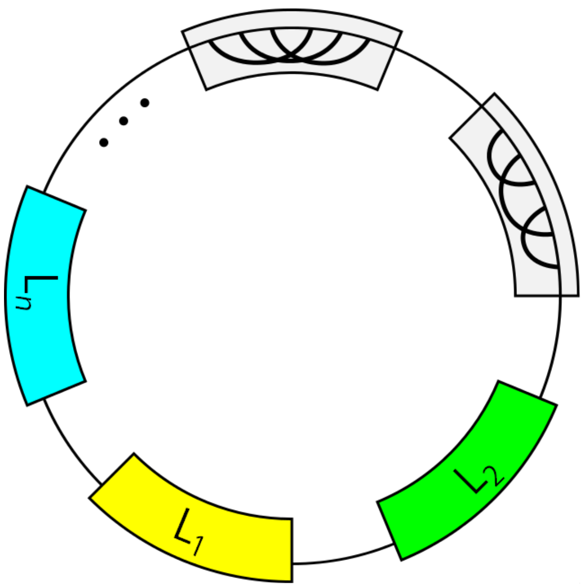


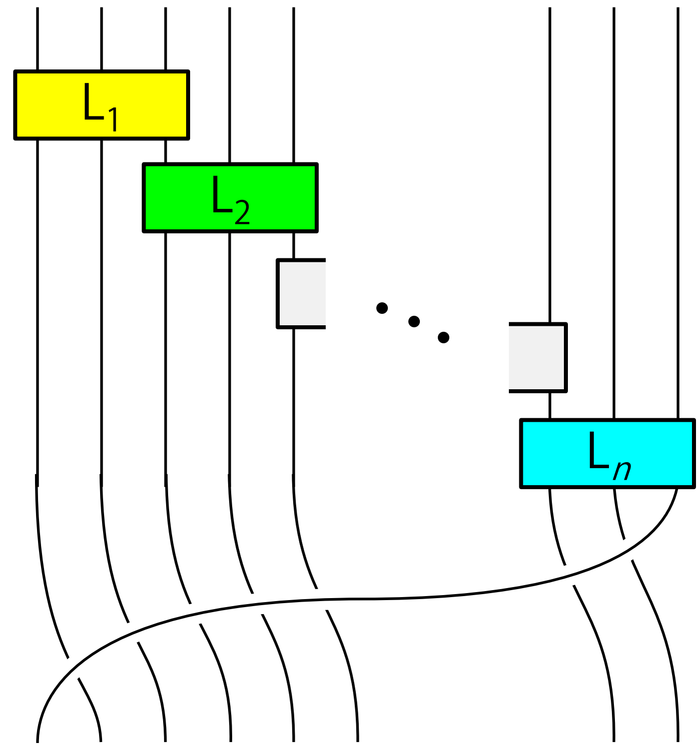
Divide and Conquer

Theorem:

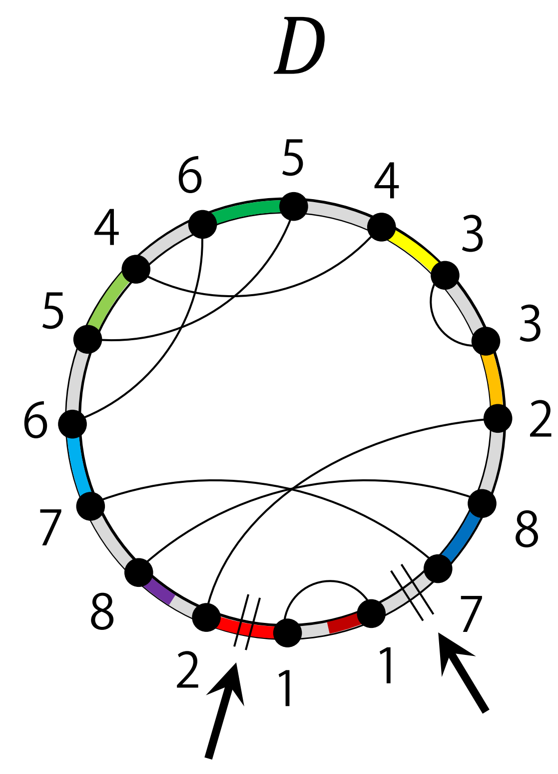
If the chord diagram D is the composition of linear chord diagrams $L_1, L_2, \ldots, L_n$ then

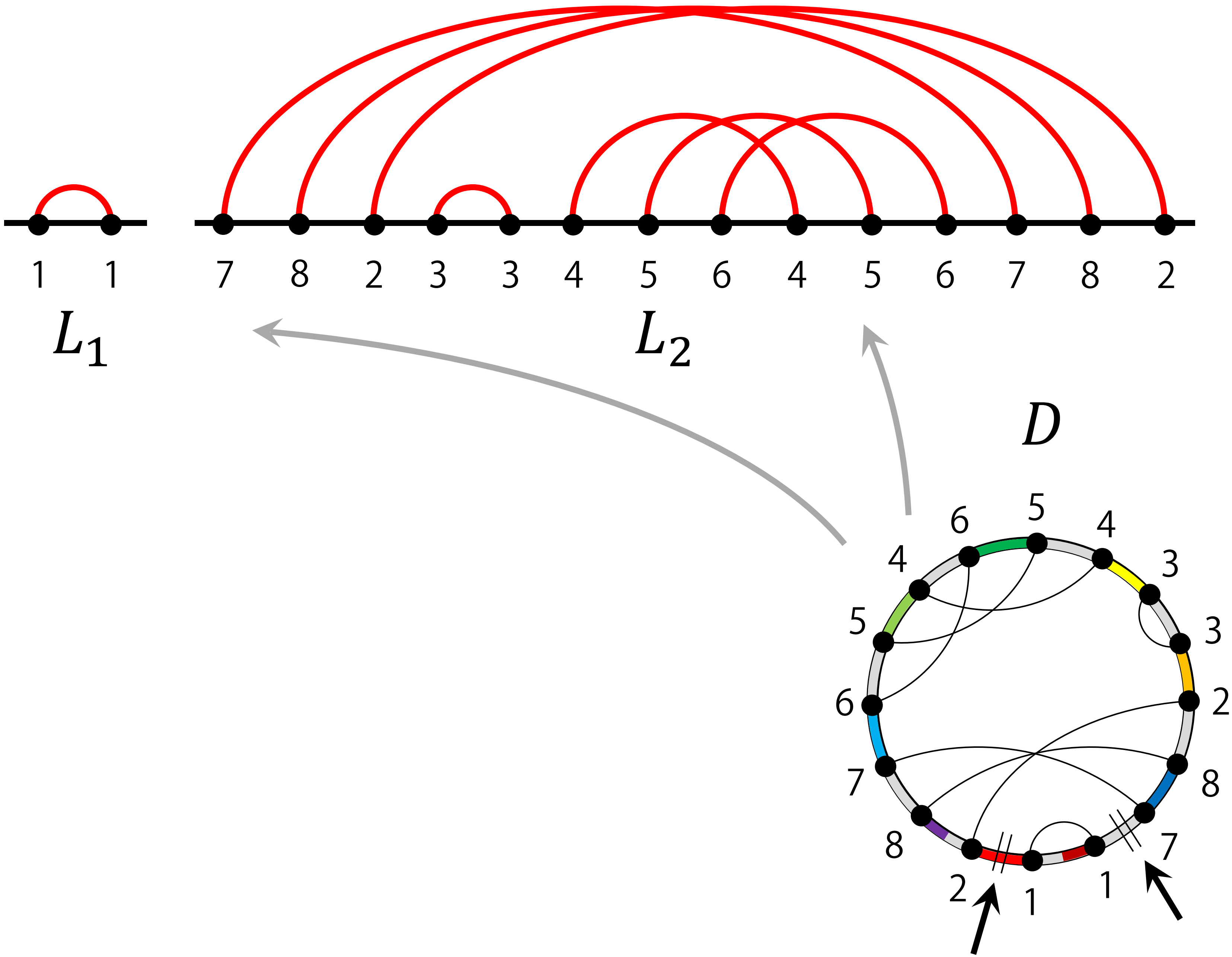
$$b(D) = \sum_{i=1}^{n} b(\overline{L_i}) - (n-1).$$



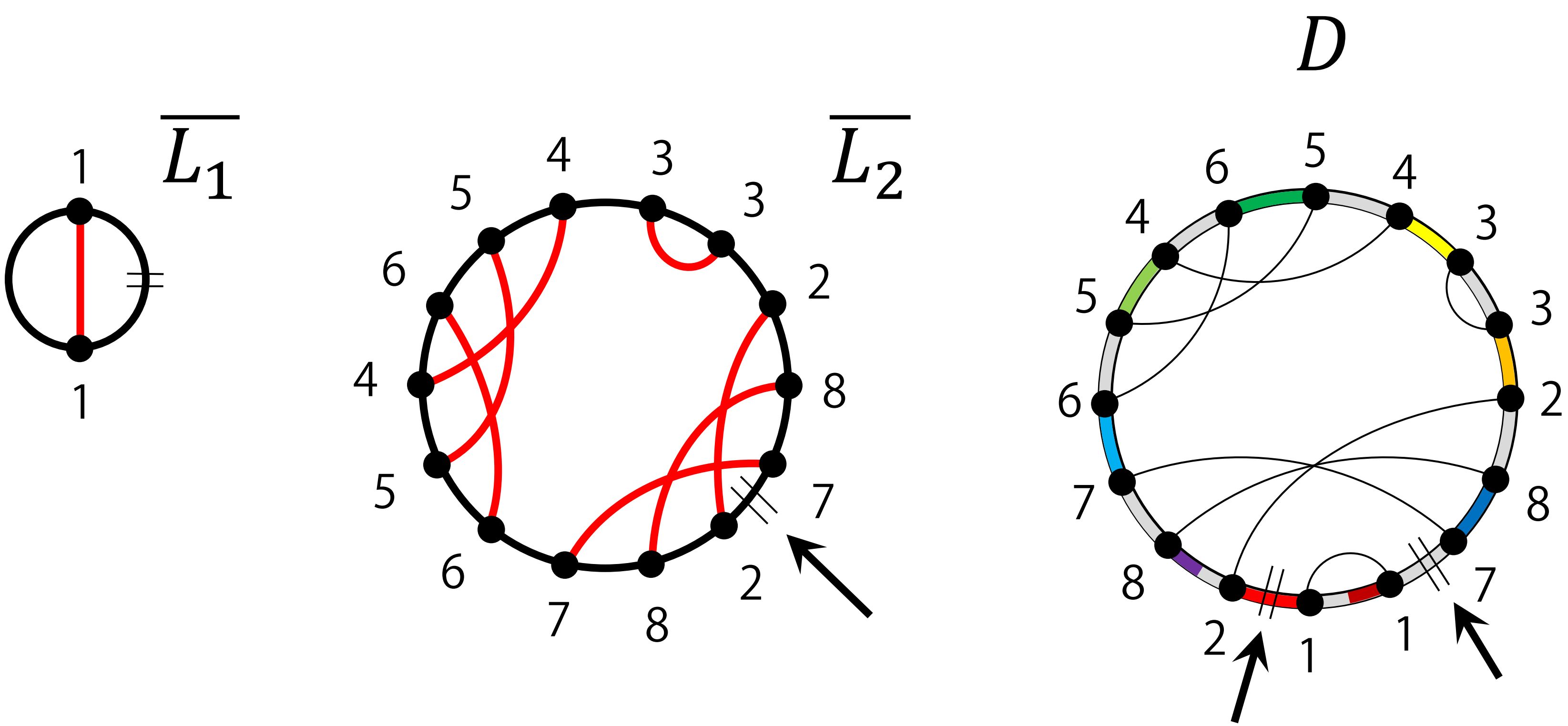


Example

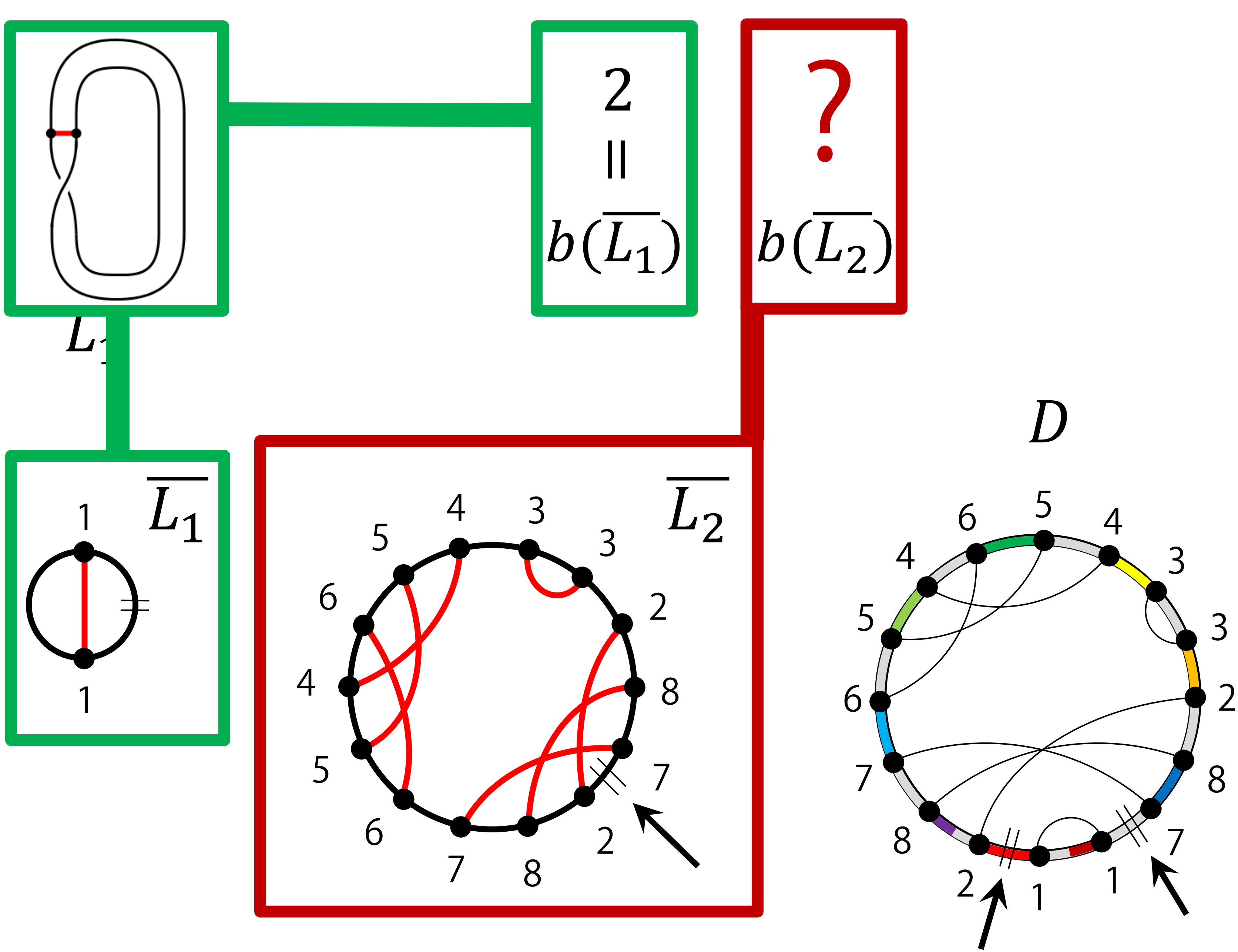




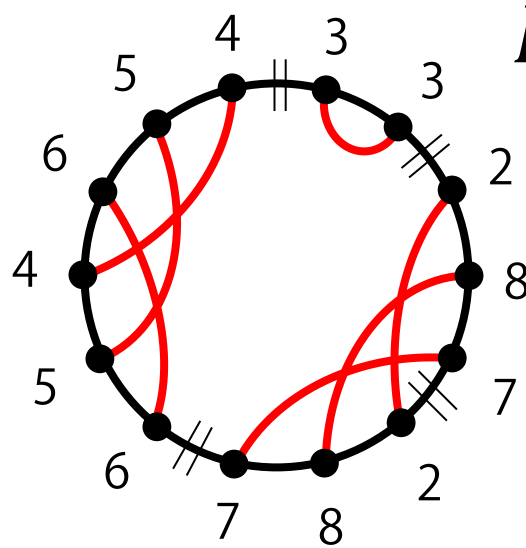
1 1
7 8 2 3 3 4 5 6 4 5 6 7 8 2
L_1
L_2
D
6 5 4
4 3
5 3
6 2
7 8
8 7
2 1 1



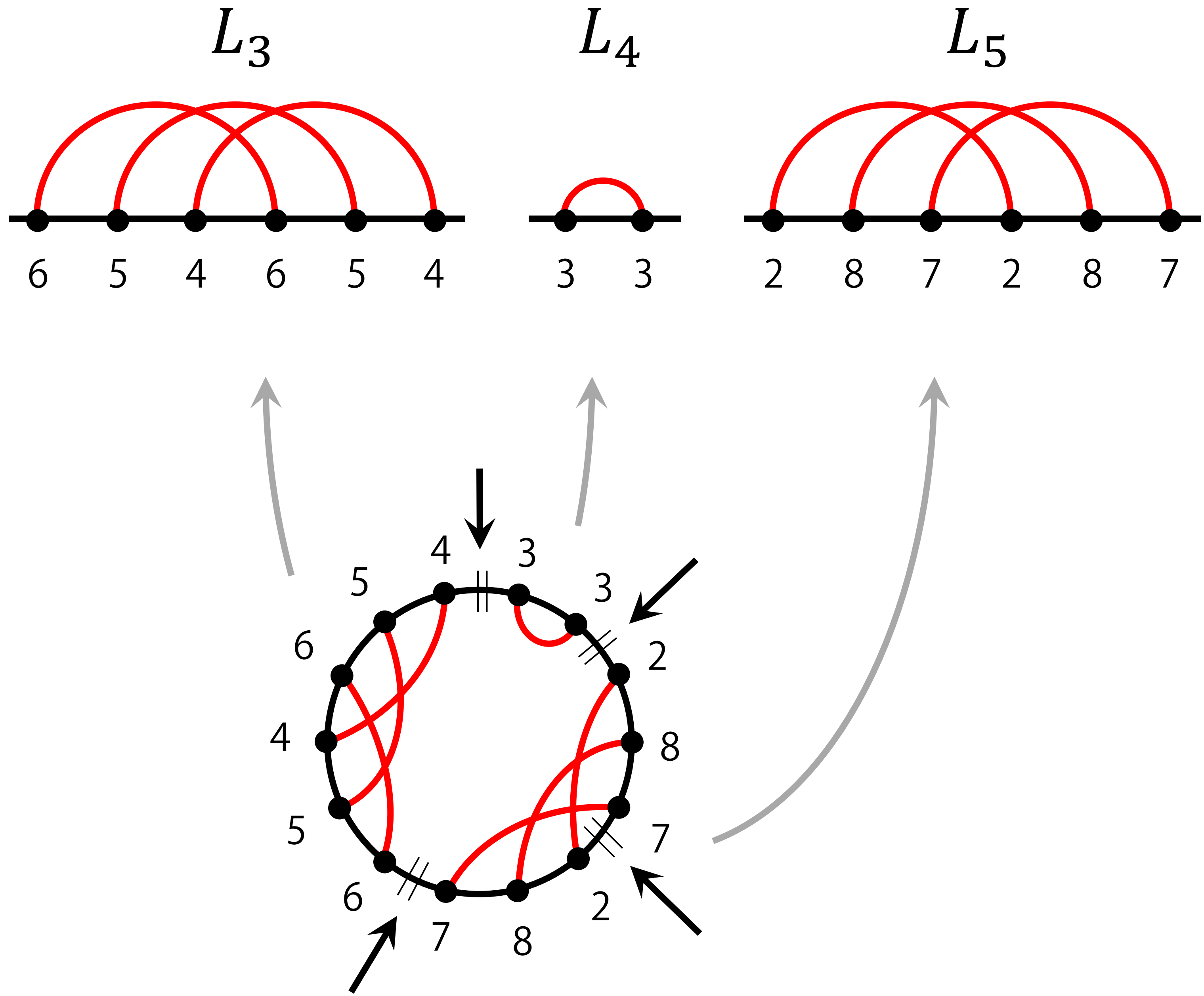
$\overline{L_1}$
1
1
$\overline{L_2}$
4
3
5
3
6
2
4
8
5
7
6
2
7
8
D
6
5
4
4
3
5
3
6
2
7
8
8
7
2
1
1



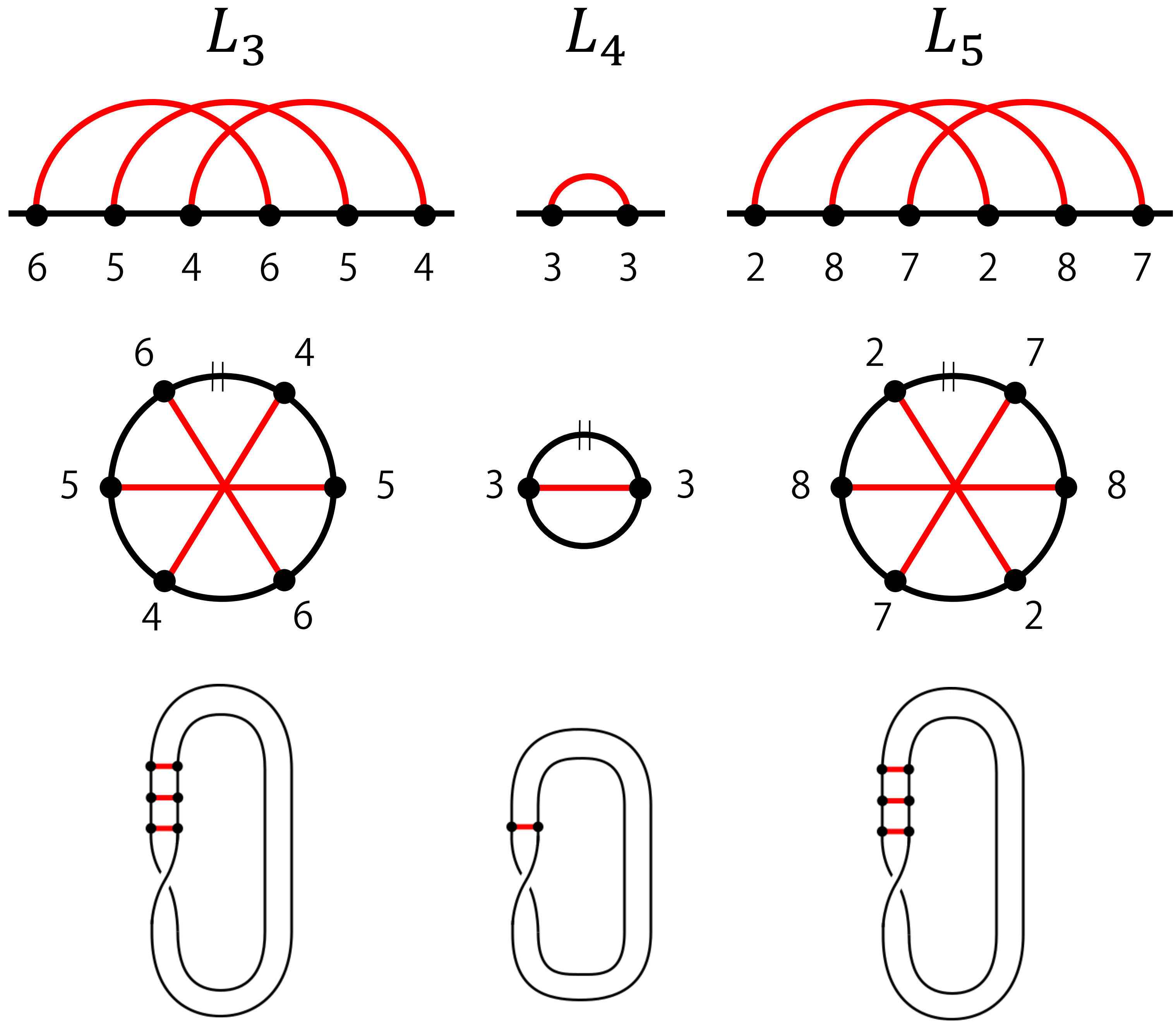
2
||
$b(\overline{L_1})$
?
$b(\overline{L_2})$
L_1
$\overline{L_1}$
1
1
$\overline{L_2}$
4 3 3 2 8 7 2 8 7 6 5 4 6 5
D
5 4 3 3 2 8 7 1 1 2 8 7 6 5 4 6



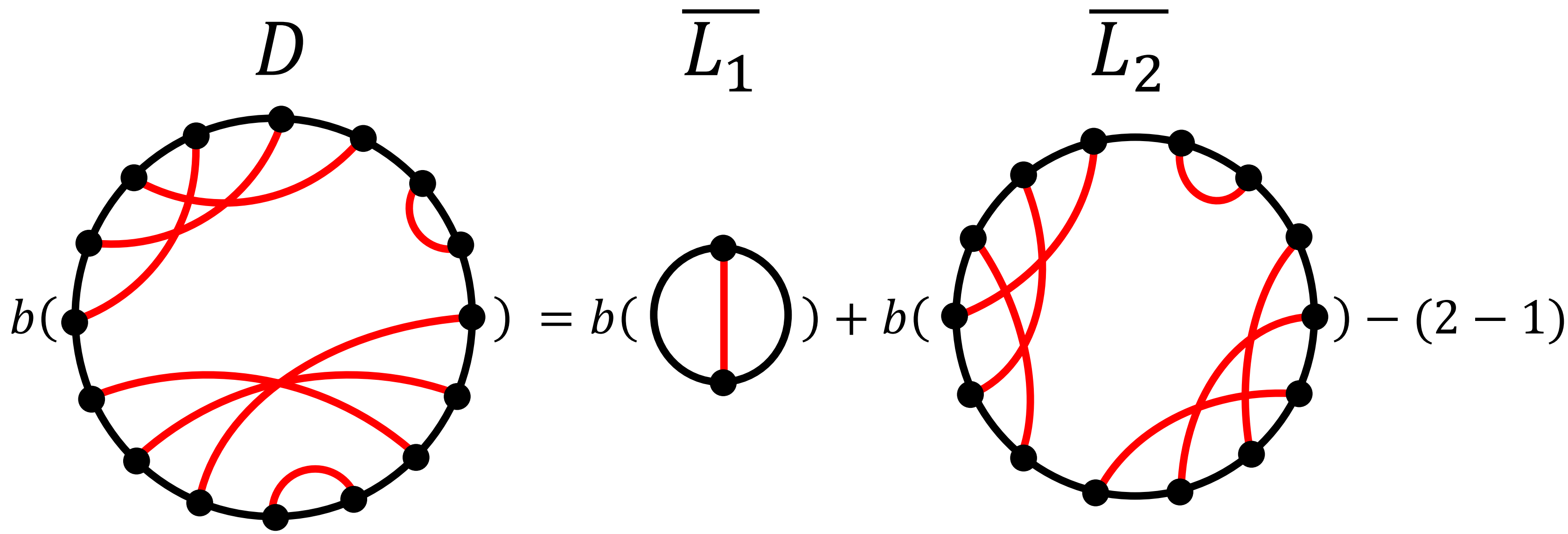
$\overline{L_2}$

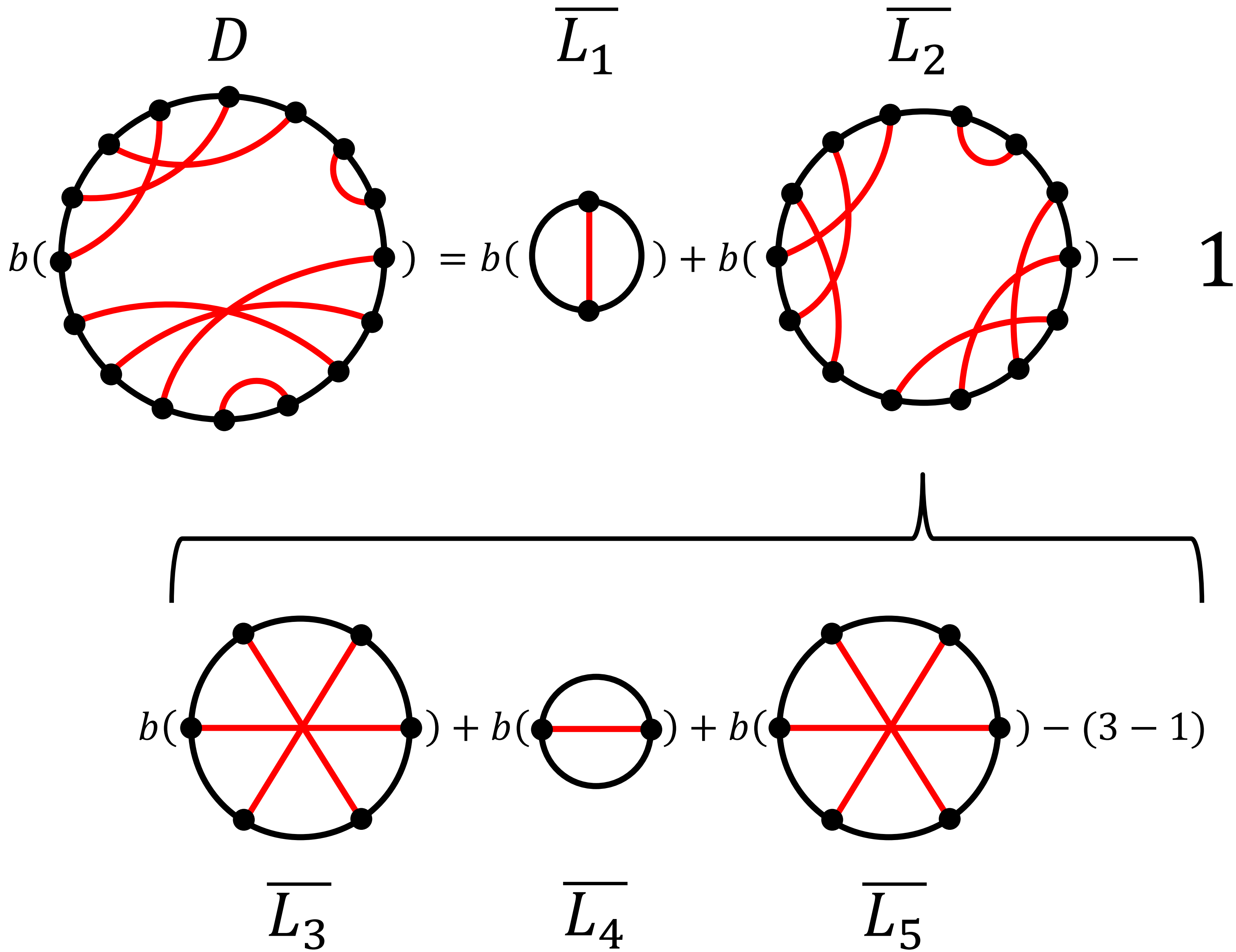


L_3
L_4
L_5
6 5 4 6 5 4
3 3
2 8 7 2 8 7
4 3 3 2 8 7 2 8 7 6 5 4 6 5

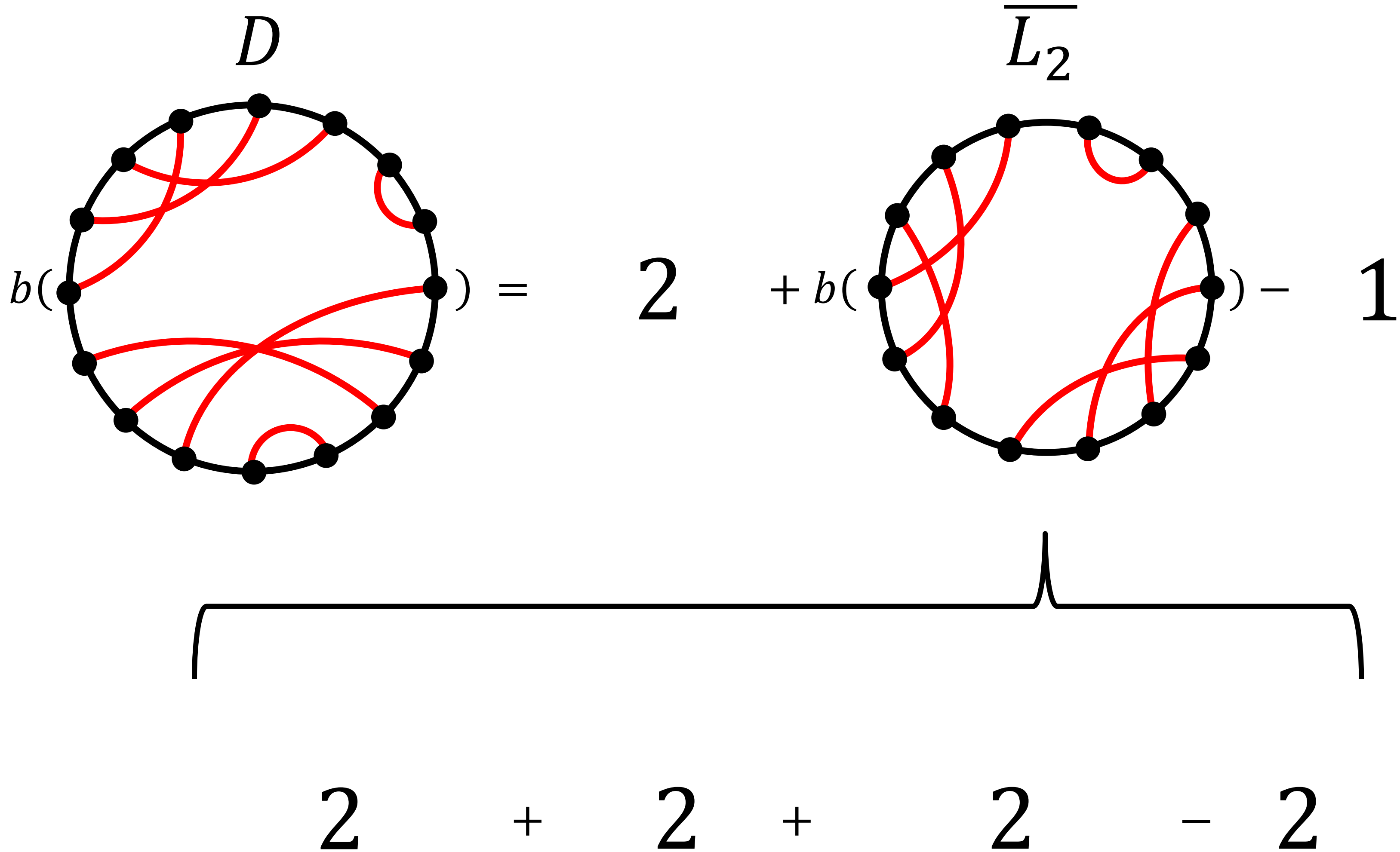


L_3
L_4
L_5
6 5 4 6 5 4
3 3
2 8 7 2 8 7
6 4
5 5
4 6
3 3
2 7
8 8
7 2

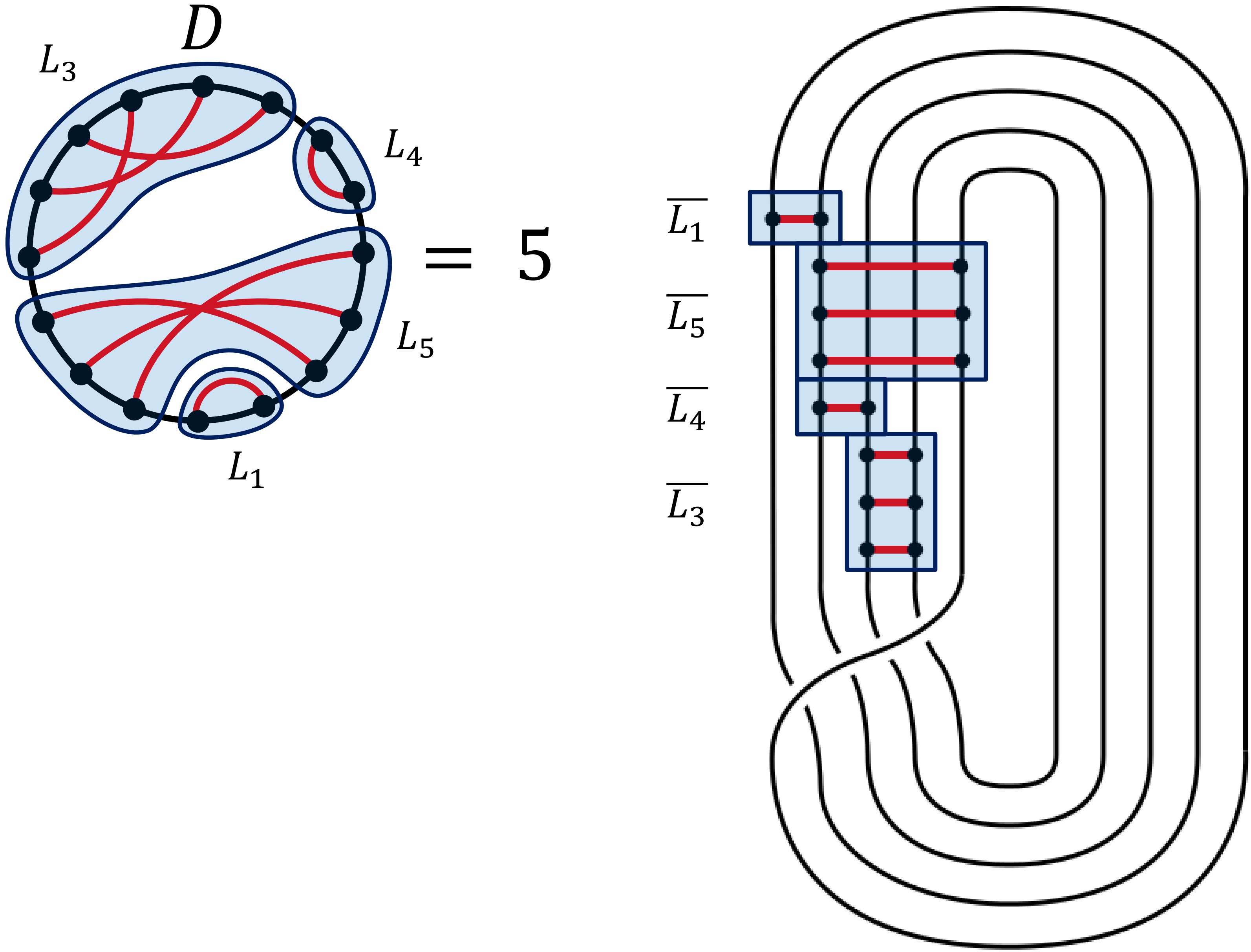

D
$\overline{L_1}$
$\overline{L_2}$
$b(\ ) = b(\ ) + b(\ ) - (2-1)$



D
$\overline{L_1}$
$\overline{L_2}$
$b(\ ) = b(\ ) + b(\ ) - 1$
$b(\ ) + b(\ ) + b(\ ) - (3 - 1)$
$\overline{L_3}$
$\overline{L_4}$
$\overline{L_5}$



D
$\overline{L_2}$
$b($ $) = 2 + b($ $) - 1$
$2 + 2 + 2 - 2$

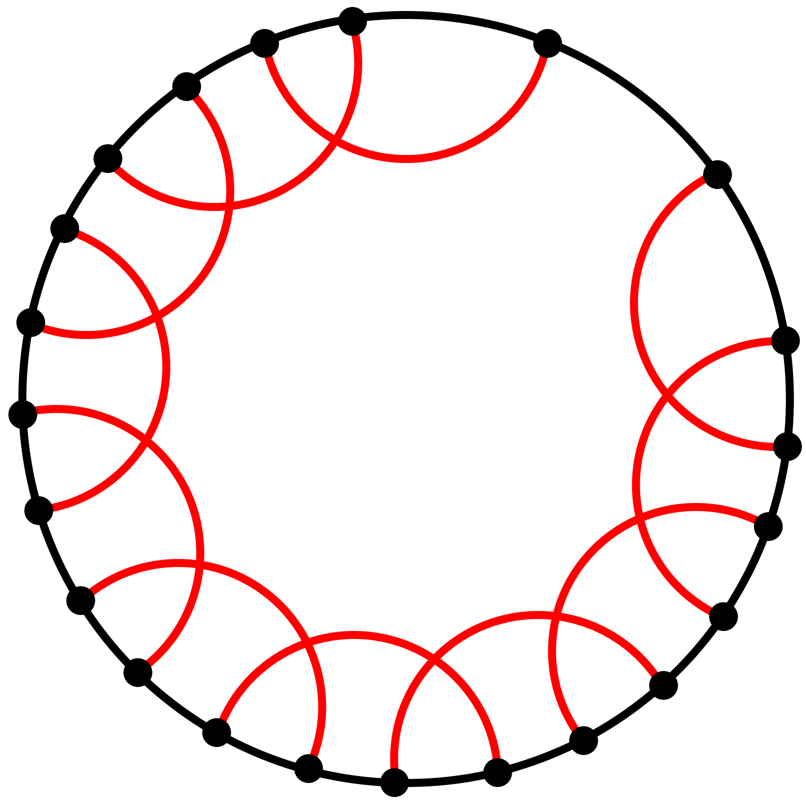


D
L_3
L_4
L_5
L_1
= 5
$\overline{L_1}$
$\overline{L_5}$
$\overline{L_4}$
$\overline{L_3}$

Isolated Chords

Corollary:

If the set of chords of the diagram D has a subset p of isolated chords, then $b(D) \geq p + 1$.



Birman & Trapp, 1998

Corollary:

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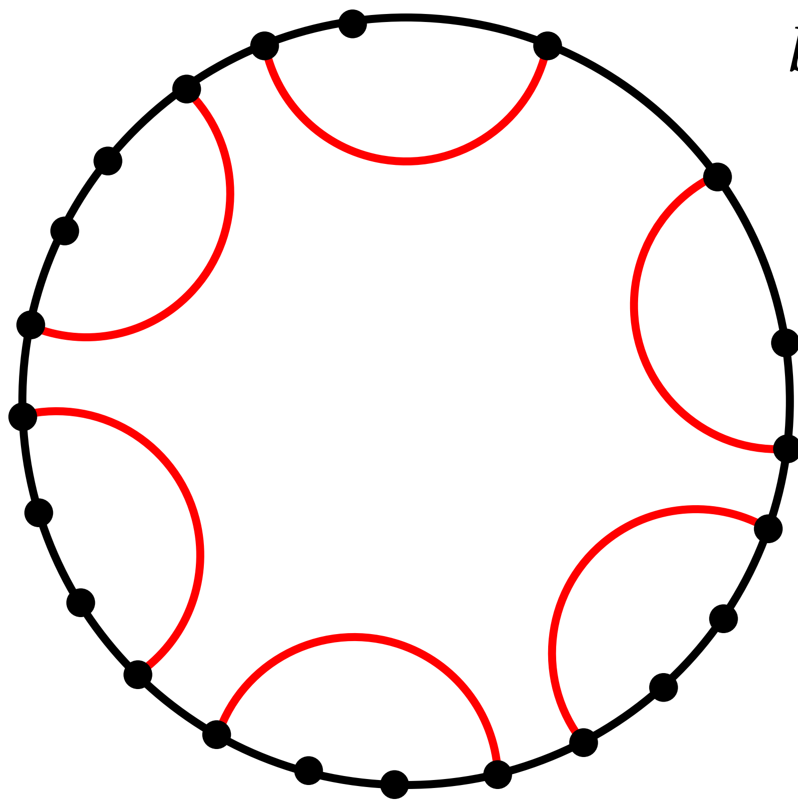
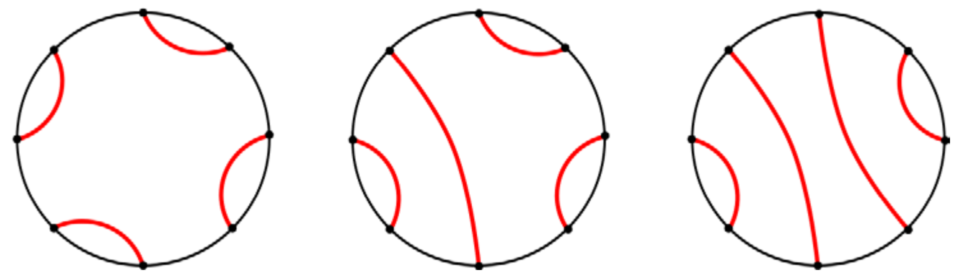


$b(D) \geq 7$

Birman & Trapp, 1998

Proposition:

A CD having n chords has braid index $n+1$ iff it contains only isolated chords.



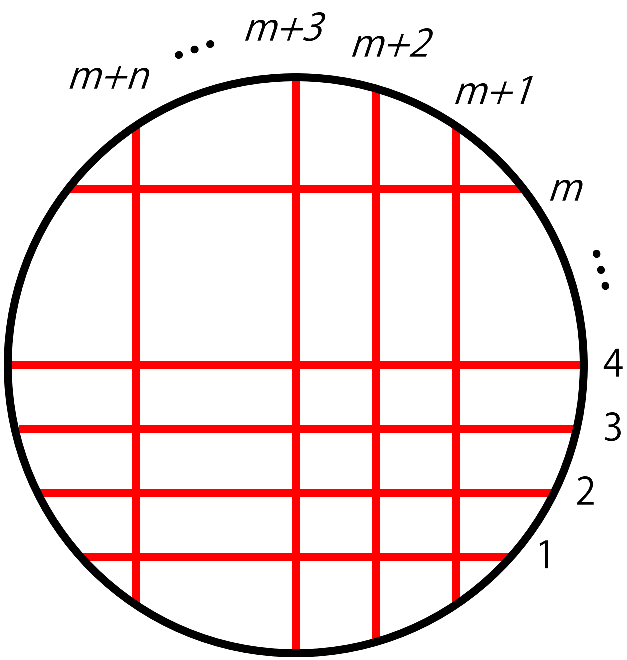
Corollary:

The number of LCDs with n chords that have braid index $n+1$ is

$$\frac{1}{n+1}\binom{2n}{n},$$

i.e., the n^{th} Catalan number.

Waffle Diagrams



$W_{m,n}$

1. *Low* Number of Isolated Chords
   – min $\{m,n\}$

2. *High* Braid Index
   – <u>Conjecture</u>:
   $$b(W_{m,n}) = m + n - 1$$

References

Journal Articles:

- J. S. Birman and R. Trapp, "Braided Chord Diagrams", *J. Knot Theory Ramifications*. **7** (1998) pp. 1-22. doi: 10.1142/S0218216598000024
- D. M. Prescott and A. F. Greslin, "Scrambled actin I gene in the micronucleus of Oxytricha nova", *Developmental Genetics*. **13**:1 (1992), pp. 66-74. doi: 10.1002/dvg.1020130111

Web Resources:

- Mathematica Code & Tables: http://jtburns.myweb.usf.edu
- LCD Search: http://knot.math.usf.edu/data/search.html

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DUE-0756847

Thank You!